Almost Hausdorff Extensions

By

DARRELL WAYNE HAJEK

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA IN PARTIAL
FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA
1971

TABLE OF CONTENTS

Abstract of Dissertation Presented to the
Graduate Council of the University of Florida in Partial Fulfillment
of the Requirements for the Degree of Doctor of Philosophy

ALMOST HAUSDORFF EXTENSIONS

By

Darrell Wayne Hajek

August, 1971

Chairman: George E. Strecker
Major Department: Mathematics

This dissertation was motivated by the property of the Wallman compactification that any continuous function from a T_1 space X into a compact Hausdorff space can be extended to a continuous function on the Wallman compactification of X. In Chapter 1 we consider general properties of extensions. We define what it means for an extension $\langle Y, \varphi \rangle$ of a space X to be bidistinguishable, C-distinguishable, and relatively Hausdorff, and show that the Wallman compactification of any T_1 space X is a bidistinguishable, C-distinguishable extension of X. Furthermore, we show that the Wallman compactification of a T_3 space is relatively Hausdorff, which allows a characterization of T_3 spaces as precisely those spaces which have compact relatively Hausdorff extensions. We also show that the collection of bi-distinguishable extensions of a space is partially ordered.

In Chapter 2 we show that the Wallman compactification on a suitably restricted subcategory of the category of T_1 spaces induces

a reflection functor which, on T_4 spaces, is the same as the Stone-Čech functor.

In Chapter 3 we topologize $\mathcal{O}(X)$, the collection of all filters in the lattice of open subsets of X. $\mathcal{O}(X)$ contains as a subspace the homeomorphic image of each C-distinguishable extension of X, and this property is used in establishing that the direct limit of a system of Hausdorff spaces is Hausdorff if each bonding map is a dense embedding.

INTRODUCTION

This study of extensions of topological spaces was originally motivated by the property of the Wallman compactification (like that of the Stone-Čech compactification for $T_{3\frac{1}{2}}$ spaces) that any continuous function from a T_1 space X into a closed unit interval (and hence into a compact Hausdorff space) can be extended to a continuous function on the Wallman compactification of X [5, p. 167]. This led to the conjecture that the Wallman compactification might well do for T_1 spaces what the Stone-Čech compactification does for completely regular T_1 spaces (i.e., it might induce a reflection functor from the category of all T_1 spaces to the category of all compact T_1 spaces). That this is true, provided one's attention is restricted to a class of functions somewhat smaller than the class of all continuous functions, is shown in Chapter 2 (2.26). But the search has led down some rather strange byways. It early became apparent that the structure of a T_1 compactification of a space X need have relatively little relationship to the structure of X. In an attempt to remove some of the pathology from this situation, we have developed four definitions of particular extension properties. These properties serve to insure that the structure of the extension has at least some relationship to the

structure of X. The definitions may be loosely described as follows:

An extension of X is distinguishable if by looking at the restriction of neighborhood families to X one can tell that the extension is T_0.

An extension of X is bidistinguishable if by looking at the restriction of neighborhood families to X one can tell that the extension is T_1.

An extension of X is C-distinguishable if by looking at the restriction of neighborhood families to X one can tell which points are in which closed sets.

An extension of X is called relatively Hausdorff if any point of X and any other point of the extension can be separated by disjoint open sets.

In Chapter 1 these concepts are precisely defined and some of their properties, as well as relationships among them, are studied. It is shown that all four concepts are preserved under products of extensions (1.15) and that continuous functions fixing X partially order the bidistinguishable extensions of X (1.20). The Wallman compactification of X is shown to be bidistinguishable and C-distinguishable (1.8) and (if X is T_3) relatively Hausdorff (1.9). This last result is then used to characterize T_3 spaces as precisely those spaces which have relatively Hausdorff compactifications (1.10).

In Chapter 2 we establish the Wallman compactification as a functor and show that it yields a reflection on a category which

contains (as a full subcategory) the category of all T_4 spaces and continuous functions (on which it is well known that the Wallman compactification and the Stone-Čech compactification coincide). Hence, the Wallman compactification on this category of T_1 spaces is precisely analogous to the Stone-Čech compactification on the category of completely regular Hausdorff spaces.

It is known that a topological direct limit of completely normal Hausdorff spaces can be an indiscrete space [2, p. 422]. However, if the spectrum of spaces forms a sequence with each space a closed subspace of all those that follow it, it is easy to establish that the direct limit will be T_1 (respectively, T_4) if each of the spaces is. That the Hausdorff property is not preserved, even under such favorable conditions, has recently been shown by Herrlich [3]. In Chapter 3 we devise sets of conditions sufficient for the preservation of the Hausdorff and regular Hausdorff properties (3.10 and 3.15). Our technique for proving the conditions sufficient will utilize the construction (for any space X) of a space $\mathcal{O}(X)$ of <u>all filters</u> in the lattice $\mathcal{J}_X$ of open sets in X. This construction is analogous to that of Strauss [6] who topologized the collection of <u>ultrafilters</u> in $\mathcal{J}_X$. $\mathcal{O}(X)$ is proved to be a C-distinguishable extension of X, and indeed it is shown that all C-distinguishable extensions of X can be considered to be precisely the subspaces of $\mathcal{O}(X)$ that contain X. This fact and a convenient mapping coherence property (3.9) are utilized to achieve the direct limit results.

CHAPTER 0

PRELIMINARIES

The notation and terminology in this dissertation will, except where noted, follow that of Kelley [5] and, when discussing categories, that of Herrlich and Strecker [4]. Unless there seems some possible confusion, a topological space $(X, \mathcal{T})$ will usually be designated by X. The symbols $\mathcal{T}_X$ and $\mathcal{L}_X$ will always designate the topology (i.e., the collection of all open sets) and the lattice of all closed subsets of X, respectively. The remainder of this chapter consists of definitions and results which are, for the most part, well known. They are included for the purpose of reference and completeness.

Definition 0.1

A function $f: X \longrightarrow Y$ will be called open if and only if, for each open subset U of X, $f[U]$ is an open subset of Y. It will be called relatively open provided that for any open subset U of X there is some open subset V of Y such that $f[U] = f[X] \cap V$.

Definition 0.2

A function $f: X \longrightarrow Y$ will be called closed provided that for each $A \in \mathcal{L}_X$, $f[A] \in \mathcal{L}_Y$.

Proposition 0.3

If $f:X \longrightarrow Y$ and $g:Y \longrightarrow Z$ are relatively open one-to-one functions, then $g\circ f:X \longrightarrow Z$ is relatively open.

Proof: Let U be an open subset of X. $f[U] = f[X] \cap V$ for some open subset V of Y and $g[V] = g[Y] \cap W$ for some open subset W of Z. Then $g\circ f[U]$ is clearly a subset of $W \cap g\circ f[X]$. For any $x \in X \sim U$, since f is one-to-one, $f[U] \neq f[U \cup \{x\}]$; so $f(x) \notin V$. Since g is one-to-one, $g[V] \neq g[V \cup \{f(x)\}]$; so $g\circ f(x) \notin W$.

Definition 0.4

A function $f:X \longrightarrow Y$ will be called an embedding provided that f is one-to-one, continuous and relatively open.

Proposition 0.5

If $f:X \longrightarrow Y$ and $g:Y \longrightarrow Z$ are embeddings, then $g\circ f:X \longrightarrow Z$ is an embedding.

Proof: Since it is well known that the composition of continuous functions is continuous, and that the composition of one-to-one functions is one-to-one, this is immediate from Proposition 0.3.

Definition 0.6

A function $f:X \longrightarrow Y$ is called dense provided that $f[X]$ is a dense subset of Y.

Proposition 0.7

If $f:X \longrightarrow Y$ and $g:Y \longrightarrow Z$ are dense continuous functions, then $g\circ f$ is dense.

Proof: Let U be any nonempty open subset of Z. Since $g[Y]$ is dense in Z, $g[Y] \cap U \neq \emptyset$; so since g is continuous, $g^{-1}[U]$ is a nonempty open subset of Y. $f[X]$ is dense in Y, so $f[X] \cap g^{-1}[U]$ is nonempty. Hence $g\circ f[X] \cap U$ is nonempty.

Definition 0.8

An extension of a space X will be a pair $\langle Y, \varphi \rangle$, where Y is a space and φ is a dense embedding of X into Y.

Notation 0.9

If Λ is a set and if for each $\lambda \in \Lambda$, X_λ and Y_λ are spaces and $f_\lambda : X_\lambda \longrightarrow Y_\lambda$ is a function, then let $\prod_{\lambda \in \Lambda} f_\lambda$ denote the function from the product $\prod_{\lambda \in \Lambda} X_\lambda$ to $\prod_{\lambda \in \Lambda} Y_\lambda$ which carries each element $(x_\lambda)_{\lambda \in \Lambda}$ of $\prod_{\lambda \in \Lambda} X_\lambda$ to the element $(f_\lambda(x_\lambda))_{\lambda \in \Lambda}$ of $\prod_{\lambda \in \Lambda} Y_\lambda$.

For Definition 0.10 through Proposition 0.17 we will assume that $(\mathcal{L}, \vee, \wedge)$ is a distributive lattice with a zero element Z.

Definition 0.10

A nonempty subset $\mathcal{F}$ of $\mathcal{L}$ will be called a filter in $\mathcal{L}$ provided that:

(i) $Z \notin \mathcal{F}$

(ii) $A, B \in \mathcal{F} \Longrightarrow A \wedge B \in \mathcal{F}$ and

(iii) $A \in \mathcal{F}$, $A \wedge B = A \Longrightarrow B \in \mathcal{F}$.

Definition 0.11

A filter $\mathcal{F}$ in $\mathcal{L}$ will be called an ultrafilter in $\mathcal{L}$ provided that $\mathcal{F}$ is not a proper subset of any filter in $\mathcal{L}$.

Proposition 0.12

Every filter $\mathcal{F}$ in $\mathcal{L}$ is contained in some ultrafilter.

Proof: The collection of all filters in $\mathcal{L}$ which contain $\mathcal{F}$ is certainly nonempty since it contains $\mathcal{F}$ and it is partially ordered by inclusion. Hence, by the Hausdorff maximality principle, it contains a maximal totally ordered subset $\mathcal{C}$. Let $\mathcal{U} = \cup\{\mathcal{G} : \mathcal{G} \in \mathcal{C}\}$. $\mathcal{U}$ is a filter containing $\mathcal{F}$ since every

element of $\mathcal{F}$ is an element of each $\mathcal{B} \in \mathcal{C}$ and:

(i) $Z \in \mathcal{U}$ implies there is some $\mathcal{B} \in \mathcal{C}$ such that $Z \in \mathcal{B}$; so $\mathcal{B}$ cannot be a filter: a contradiction.

(ii) Given any A and B in $\mathcal{U}$ there is some $\mathcal{B}_1 \in \mathcal{C}$ such that $A \in \mathcal{B}_1$ and some $\mathcal{B}_2 \in \mathcal{C}$ such that $B \in \mathcal{B}_2$. Since $\mathcal{C}$ is totally ordered by inclusion, one must contain the other (say $\mathcal{B}_1 \subseteq \mathcal{B}_2$). Then both A and B are elements of $\mathcal{B}_2$ so $A \wedge B \in \mathcal{B}_2$. Hence $A \wedge B \in \mathcal{U}$.

(iii) If $A \in \mathcal{U}$ and $B \wedge A = A$, then $A \in \mathcal{B}$ for some $\mathcal{B} \in \mathcal{C}$, and since $\mathcal{B}$ is a filter, $B \in \mathcal{B}$. Hence $B \in \mathcal{U}$.

If $\mathcal{U}$ is not an ultrafilter, then $\mathcal{U}$ is a proper subset of some filter $\mathcal{U}'$. Since $\mathcal{F} \subseteq \mathcal{U}$ and $\mathcal{U} \subseteq \mathcal{U}'$, $\mathcal{F} \subseteq \mathcal{U}'$; so $\mathcal{U}'$ is in the collection of all filters containing $\mathcal{F}$. For each $\mathcal{B} \in \mathcal{C}$, $\mathcal{B} \subseteq \mathcal{U} \subseteq \mathcal{U}'$; so $\mathcal{C} \cup \{\mathcal{U}'\}$ is totally ordered by inclusion and properly contains $\mathcal{C}$. This contradicts the maximality of $\mathcal{C}$; so $\mathcal{U}$ is an ultrafilter.

<u>Proposition 0.13</u>

If $\mathcal{A} \subseteq \mathcal{L}$ and if for every finite subset $\mathcal{B}$ of $\mathcal{A}$, $\wedge\mathcal{B} \neq Z$, then there is a unique smallest filter $\mathcal{F}_{\mathcal{A}}$ in $\mathcal{L}$ which contains $\mathcal{A}$.

<u>Proof</u>: Let $\mathcal{A}' = \{\wedge\mathcal{B} : \mathcal{B}$ a finite subset of $\mathcal{A}\}$. Let $\mathcal{F}_{\mathcal{A}} = \{B \in \mathcal{L} : B \wedge D = D$ for some $D \in \mathcal{A}'\}$. $\mathcal{F}_{\mathcal{A}}$ is a filter in $\mathcal{L}$ because:

(i) If $Z \in \mathcal{F}_{\mathcal{A}}$, then there is some $D \in \mathcal{A}'$ such that $Z = Z \wedge D = D$; but this contradicts the assumption about $\mathcal{A}$.

(ii) If B_1 and B_2 are elements of $\mathcal{F}_{\mathcal{A}}$, then there are $\wedge\mathcal{B}_1$ and $\wedge\mathcal{B}_2$ in $\mathcal{A}'$ such that $B_1 \wedge (\wedge\mathcal{B}_1) = \wedge\mathcal{B}_1$ and $B_2 \wedge (\wedge\mathcal{B}_2) = \wedge\mathcal{B}_2$. Then $(\wedge\mathcal{B}_1) \wedge (\wedge\mathcal{B}_2) = \wedge(\mathcal{B}_1 \cup \mathcal{B}_2)$

is an element of $\mathcal{A}'$ and

$(B_1 \wedge B_2) \wedge ((\wedge \mathcal{B}_1) \wedge (\wedge \mathcal{B}_2)) = B_1 \wedge (\wedge \mathcal{B}_1)) \wedge (B_2 \wedge (\wedge \mathcal{B}_2)) = (\wedge \mathcal{B}_1) \wedge (\wedge \mathcal{B}_2)$;

so $B_1 \wedge B_2 \in \mathcal{F}_{\mathcal{A}}$.

(iii) If $B \in \mathcal{F}_{\mathcal{A}}$ and $B \wedge D = B$, there is some $E \in \mathcal{A}'$ such that $B \wedge E = E$. Then $D \wedge E = D \wedge (B \wedge E) = (D \wedge B) \wedge E = B \wedge E = E$; so $D \in \mathcal{F}_{\mathcal{A}}$.

Finally, it is clear that every element of $\mathcal{F}_{\mathcal{A}}$ must be an element of any filter containing $\mathcal{A}$.

Corollary 0.14.

If $\mathcal{A}$ is a subset of $\mathcal{L}$ such that for every finite subset $\mathcal{B}$ of $\mathcal{A}$, $\wedge \mathcal{B} \neq Z$, then there is an ultrafilter in $\mathcal{L}$ which contains $\mathcal{A}$.

Proof: This is immediate from Propositions 0.12 and 0.13.

Proposition 0.15

If $\mathcal{U}$ and $\mathcal{U}'$ are distinct ultrafilters in $\mathcal{L}$, then there is some $A \in \mathcal{U}$ and some $B \in \mathcal{U}'$ such that $A \wedge B = Z$.

Proof: If no two such elements exist, select any finite subset $\mathcal{B}$ of $\mathcal{U} \cup \mathcal{U}'$. If $\mathcal{B} \subseteq \mathcal{U}$ or $\mathcal{B} \subseteq \mathcal{U}'$, then since $\mathcal{B}$ is finite, $\wedge \mathcal{B}$ is respectively an element of $\mathcal{U}$ or $\mathcal{U}'$, and so is not Z. If $\mathcal{B} \cap \mathcal{U}$ and $\mathcal{B} \cap \mathcal{U}'$ are both nonempty, then since each is finite, $\wedge (\mathcal{B} \cap \mathcal{U})$ is an element of $\mathcal{U}$ and $\wedge (\mathcal{B} \cap \mathcal{U}')$ is an element of $\mathcal{U}'$. Hence $\wedge \mathcal{B} = (\wedge (\mathcal{B} \cap \mathcal{U})) \wedge (\wedge (\mathcal{B} \cap \mathcal{U}')) \neq Z$. Then from Proposition 0.13, $\mathcal{U} \cup \mathcal{U}'$ is contained in some filter $\mathcal{F}_{\mathcal{U} \cup \mathcal{U}'}$ in $\mathcal{L}$. Since $\mathcal{U}$ and $\mathcal{U}'$ are distinct, they are proper subsets of $\mathcal{F}_{\mathcal{U} \cup \mathcal{U}'}$, which contradicts the fact that they are ultrafilters.

Proposition 0.16

If $\mathcal{U}$ is an ultrafilter in $\mathcal{L}$ and if $A \in \mathcal{L}$ and $A \wedge B \neq Z$ for every $B \in \mathcal{U}$, then $A \in \mathcal{U}$.

Proof: If A satisfies the above conditions and is not an element of $\mathcal{U}$, then for any finite subset $\mathcal{B}$ of $\mathcal{U} \cup \{A\}, \wedge \mathcal{B} \neq Z$; so from Proposition 0.13 there is a filter $\mathcal{F}_{\mathcal{U} \cup \{A\}}$ which contains $\mathcal{U}$ as a proper subset. This contradicts the assumption that $\mathcal{U}$ is an ultrafilter.

Proposition 0.17

If $\mathcal{U}$ is an ultrafilter in $\mathcal{L}$ and if $A \vee B \in \mathcal{U}$, then $A \in \mathcal{U}$ or $B \in \mathcal{U}$.

Proof: If $B \notin \mathcal{U}$, then from Proposition 0.16 there is some $D \in \mathcal{U}$ such that $B \wedge D = Z$. Then

$D \wedge (A \vee B) = (D \wedge A) \vee (D \wedge B) = (D \wedge A) \vee Z = D \wedge A \in \mathcal{U}$.

Then since $A \wedge (D \wedge A) = D \wedge (A \wedge A) = D \wedge A$, A must be an element of $\mathcal{U}$ [0.10(iii)].

Definition 0.18

If X and Y are sets and R is a subset of $X \times Y$, then for any subset A of X, we define $R[A]$ to be $\{y \in Y: (x,y) \in R \text{ for some } x \in A\}$.

Proposition 0.19

If X, Y, and R are as in Definition 0.18 and if A and B are subsets of X, then $R[A \cap B] \subseteq R[A] \cap R[B]$.

Proof: If $y \in R[A \cap B]$, then there is some $x \in A \cap B$ such that $(x,y) \in R$. This implies that $x \in A$ and $x \in B$, hence $y \in R[A]$ and $y \in R[B]$; so $y \in R[A] \cap R[B]$.

Definition 0.20

For any set X a collection $\mathcal{L}$ of subsets of X will be called a standard lattice of subsets of X provided that $\emptyset \in \mathcal{L}$ and $X \in \mathcal{L}$

and for any two elements A and B of $\mathcal{L}$, both $A \cup B$ and $A \cap B$ are elements of $\mathcal{L}$.

<u>Definition 0.21</u>

Let X and Y be sets, let R be a subset of $X \times Y$, and let $\mathcal{L}$ and $\mathcal{L}'$ be standard lattices of subsets of X and Y, respectively, having the property that for each $A \in \mathcal{L}$, $R[A] = \emptyset$ only if $A = \emptyset$. Then for any filter $\mathcal{F}$ in $\mathcal{L}$, define $F_R(\mathcal{F})$ to be $\{A \in \mathcal{L}' : R[B] \subseteq A$ for some $B \in \mathcal{F}\}$.

<u>Proposition 0.22</u>

$F_R(\mathcal{F})$ (as defined above) is a filter in $\mathcal{L}'$.

<u>Proof</u>: $F_R(\mathcal{F})$ is a nonempty set since $X \in \mathcal{F}$ and $R[X] \subseteq Y \in \mathcal{L}'$ implies $Y \in F_R(\mathcal{F})$.

(i) If $\emptyset \in F_R(\mathcal{F})$, then there is some $A \in \mathcal{F}$ such that $R[A] \subseteq \emptyset$. But A, being in $\mathcal{F}$, is nonempty; so that this is impossible.

(ii) If A and B are elements of $F_R(\mathcal{F})$, then there are some sets D and E in $\mathcal{F}$ such that $R[D] \subseteq A$ and $R[E] \subseteq B$. Then $D \cap E \in \mathcal{F}$ and from Proposition 0.19, $R[D \cap E] \subseteq R[D] \cap R[E] \subseteq A \cap B$. Hence, $A \cap B \in F_R(\mathcal{F})$.

(iii) If $A \in F_R(\mathcal{F})$ and $A \subseteq B \in \mathcal{L}'$, then there is some set $D \in \mathcal{F}$ such that $R[D] \subseteq A$. Hence $R[D] \subseteq B$, which implies that $B \in F_R(\mathcal{F})$.

For Definition 0.23 through Proposition 0.30, we will assume that $\mathcal{L}$ is a distributive lattice with zero element $\emptyset$ and unit element X.

<u>Definition 0.23</u>

The <u>Wallman set for $\mathcal{L}$</u> (denoted $\mathcal{W}(\mathcal{L})$) is the collection of all ultrafilters in $\mathcal{L}$.

Definition 0.24

For any $M \in \mathcal{L}$, define $C(M)$ to be $\{u \in \mathcal{U}(\mathcal{L}) : M \in u\}$.

Proposition 0.25

If M_1 and M_2 are elements of $\mathcal{L}$, then $C(M_1 \vee M_2) = C(M_1) \cup C(M_2)$.

Proof: If $u \in C(M_1 \vee M_2)$, then $M_1 \vee M_2 \in u$, so from Proposition 0.17, $M_1 \in u$ or $M_2 \in u$. Hence $u \in C(M_1)$ or $u \in C(M_2)$, i.e., $u \in C(M_1) \cup C(M_2)$. If $u \in C(M_1) \cup C(M_2)$, then $u \in C(M_1)$ or $u \in C(M_2)$ (say, $u \in C(M_1)$). Because $u \in C(M_1)$, $M_1 \in u$; so since u is a filter and $M_1 \wedge (M_1 \vee M_2) = M_1$, $M_1 \vee M_2 \in u$. Hence $u \in C(M_1 \vee M_2)$.

Proposition 0.26

If M_1 and M_2 are elements of $\mathcal{L}$, then $C(M_1 \wedge M_2) = C(M_1) \cap C(M_2)$.

Proof: If $u \in C(M_1 \wedge M_2)$, then $M_1 \wedge M_2 \in u$. Since $(M_1 \wedge M_2) \wedge M_1 = M_1 \wedge M_2$, $(M_1 \wedge M_2) \wedge M_2 = M_1 \wedge M_2$, and u is a filter, $M_1 \in u$ and $M_2 \in u$. Then $u \in C(M_1)$ and $u \in C(M_2)$; so $u \in C(M_1) \cap C(M_2)$. Conversely, if $u \in C(M_1) \cap C(M_2)$, then $u \in C(M_1)$ and $u \in C(M_2)$; so $M_1 \in u$ and $M_2 \in u$. Because u is a filter, $M_1 \wedge M_2 \in u$; so $u \in C(M_1 \wedge M_2)$.

Proposition 0.27

$C(M) = \emptyset$ if and only if $M = \emptyset$.

Proof: If $u \in C(\emptyset)$, then $\emptyset \in u$, which contradicts the fact that u is a filter in $\mathcal{L}$. Hence $C(\emptyset) = \emptyset$. If M is nonempty, then M is an element of at least one ultrafilter in $\mathcal{L}$ since $\{M\}$ meets the conditions of Corollary 0.14. Hence $C(M) \neq \emptyset$.

Proposition 0.28

$\{C(M):M \in \mathcal{L}\}$ is a base for the closed sets of a topology on $\mathcal{W}(\mathcal{L})$.

Proof: This is immediate from Propositions 0.25 and 0.27.

Henceforth the symbol $\mathcal{W}(\mathcal{L})$ will designate the Wallman set on $\mathcal{L}$ with the topology of Proposition 0.28.

Proposition 0.29

$\mathcal{W}(\mathcal{L})$ is a compact T_1 space.

Proof: Given distinct elements u and v of $\mathcal{W}(\mathcal{L})$, there exist M_1 and M_2 in $\mathcal{L}$ such that $M_1 \wedge M_2 = \emptyset$, and $M_1 \in$ u and $M_2 \in$ v (0.15). Then $u \in C(M_1)$, $v \in C(M_2)$, and $C(M_1)$ and $C(M_2)$ are closed in $\mathcal{W}(\mathcal{L})$. $C(M_1)$ and $C(M_2)$ are disjoint since, from Proposition 0.26, $C(M_1) \cap C(M_2) = C(M_1 \wedge M_2)$, which is $C(\emptyset)$ since M_1 and M_2 are disjoint. From Proposition 0.27, $C(\emptyset) = \emptyset$; so $\mathcal{W}(\mathcal{L})$ is T_1. Let $\{C(M_\alpha): \alpha \in \mathcal{A}\}$ be a collection of basic closed sets with the finite intersection property. Then $\{M_\alpha : \alpha \in \mathcal{A}\}$ has the property that for any finite subset $\mathcal{B}$, $\wedge \mathcal{B} \neq \emptyset$ (0.26 and 0.27). Hence from Corollary 0.14, there is an ultrafilter u in $\mathcal{W}(\mathcal{L})$ which contains $\{M_\alpha : \alpha \in \mathcal{A}\}$. Then $u \in C(M_\alpha)$ for each $\alpha \in \mathcal{A}$, so $u \in \bigcap_{\alpha \in \mathcal{A}} C(M_\alpha)$. Therefore, $\mathcal{W}(\mathcal{L})$ is compact.

Proposition 0.30

$\mathcal{W}(\mathcal{L})$ is Hausdorff if and only if given any two elements A and B of $\mathcal{L}$ such that $A \wedge B = \emptyset$ there exist M_1 and M_2 in $\mathcal{L}$ such that $A \wedge M_1 = A$, $B \wedge M_2 = B$, $A \wedge M_2 = \emptyset$, $B \wedge M_1 = \emptyset$, and $M_1 \vee M_2$ is an element of every ultrafilter in $\mathcal{L}$.

Proof: If the condition holds and if u and v are distinct elements of $\mathcal{W}(\mathcal{L})$, then there are elements A and B of $\mathcal{L}$ such

that $A \wedge B = \emptyset$, $A \in u$, and $B \in v$ (0.15). If M_1 and M_2 are elements $\mathcal{L}$ satisfying the given conditions, then $A \wedge M_1 = A$ implies that $M_1 \in u$ and $B \wedge M_2 = B$ implies that $M_2 \in v$. Since $M_1 \wedge B = \emptyset$ and $M_2 \wedge A = \emptyset$, $M_1 \notin v$ and $M_2 \notin u$. $\mathcal{W}(\mathcal{L}) = C(M_1 \vee M_2) = C(M_1) \cup C(M_2)$ (0.25); so $\mathcal{W}(\mathcal{L}) \sim C(M_1)$ and $\mathcal{W}(\mathcal{L}) \sim C(M_2)$ are disjoint open sets in $\mathcal{W}(\mathcal{L})$ containing v and u, respectively. Conversely, if $\mathcal{W}(\mathcal{L})$ is Hausdorff, since $\mathcal{W}(\mathcal{L})$ is compact, it is normal. Hence, given any two elements A and B of $\mathcal{L}$ such that $A \wedge B = \emptyset$, $C(A)$ and $C(B)$ are disjoint closed subsets of $\mathcal{W}(\mathcal{L})$ (0.27); so there exist disjoint open sets U and V in $\mathcal{W}(\mathcal{L})$ such that $C(A) \subseteq U$ and $C(B) \subseteq V$. Then $\mathcal{W}(\mathcal{L}) \sim U$ and $\mathcal{W}(\mathcal{L}) \sim V$ are closed subsets of $\mathcal{W}(\mathcal{L})$. Since $\{C(M) : M \in \mathcal{L}\}$ is a base for the closed sets in $\mathcal{W}(\mathcal{L})$, there exist collections $\{M_\alpha : \alpha \in \mathcal{A}\}$ and $\{M_\beta : \beta \in \mathcal{B}\}$ of elements of $\mathcal{L}$ such that $\bigcap_{\alpha \in \mathcal{A}} C(M_\alpha) = \mathcal{W}(\mathcal{L}) \sim U$ and $\bigcap_{\beta \in \mathcal{B}} C(M_\beta) = \mathcal{W}(\mathcal{L}) \sim V$. Then $\{\mathcal{W}(\mathcal{L}) \sim C(M_\alpha)\}$ and $\{\mathcal{W}(\mathcal{L}) \sim C(M_\beta)\}$ are open covers of $C(A)$ and $C(B)$, respectively. $C(A)$ and $C(B)$ are closed subsets of a compact space, hence they are compact; so there exist finite subcollections $\mathcal{D} \subseteq \{M_\alpha : \alpha \in \mathcal{A}\}$ and $\mathcal{E} \subseteq \{M_\beta : \beta \in \mathcal{B}\}$ such that $\mathcal{W}(\mathcal{L}) \sim U \subseteq \bigcap_{D \in \mathcal{D}} C(D) \subseteq \mathcal{W}(\mathcal{L}) \sim C(A)$ and $\mathcal{W}(\mathcal{L}) \sim V \subseteq \bigcap_{E \in \mathcal{E}} C(E) \subseteq \mathcal{W}(\mathcal{L}) \sim C(B)$. From Proposition 0.26 $\bigcap_{D \in \mathcal{D}} C(D) = C(\wedge \mathcal{D})$ and $\bigcap_{E \in \mathcal{E}} C(E) = C(\wedge \mathcal{E})$. Hence $C(A) \subseteq \mathcal{W}(\mathcal{L}) \sim C(\wedge \mathcal{D}) \subseteq C(\wedge \mathcal{E})$ and $C(B) \subseteq \mathcal{W}(\mathcal{L}) \sim C(\wedge \mathcal{E}) \subseteq C(\wedge \mathcal{D})$. Then $A \vee (\wedge \mathcal{E})$ and $B \vee (\wedge \mathcal{D})$ are elements of $\mathcal{L}$ and $(A \vee (\wedge \mathcal{E})) \vee (B \vee (\wedge \mathcal{D}))$ is an element of $\mathcal{L}$ which is contained in every element of $\mathcal{W}(\mathcal{L})$. Now, $A \wedge (A \vee (\wedge \mathcal{E})) = A$ and $B \wedge (B \vee (\wedge \mathcal{D})) = B$. Thus, $A \wedge (B \vee (\wedge \mathcal{D})) = \emptyset \vee \emptyset = \emptyset$ and $B \wedge (A \vee (\wedge \mathcal{E})) = \emptyset \vee \emptyset = \emptyset$.

For Proposition 0.31 through Corollary 0.38, we will have the following standing hypotheses:

(i) X is a T_1 space.

(ii) $\mathcal{L}$ is a base for the closed sets in X such that for any $x \in X$ and any $B \in \mathcal{L}$ for which $x \notin B$, there is some $A \in \mathcal{L}$ such that $x \in A$ and $A \cap B = \emptyset$.

(iii) For each $x \in X$, $\varphi_{\mathcal{L}}(x) = \{A \in \mathcal{L} : x \in A\}$.

Proposition 0.31

$\varphi_{\mathcal{L}}(x)$ is an ultrafilter in $\mathcal{L}$.

Proof: $\varphi_{\mathcal{L}}(x)$ is by definition a subset of $\mathcal{L}$. Since $\{x\}$ is closed in X, it is the intersection of a collection of elements of $\mathcal{L}$; so in particular $\varphi_{\mathcal{L}}(x)$ is nonempty. $\varphi_{\mathcal{L}}(x)$ is a filter because:

(i) For every $A \in \varphi_{\mathcal{L}}(x)$, $x \in A$; so $A \neq \emptyset$.

(ii) If $x \in A$ and $x \in B$, then $x \in A \cap B$.

(iii) If $x \in A \subseteq B$, then $x \in B$.

$\varphi_{\mathcal{L}}(x)$ is an ultrafilter since if $x \notin B \in \mathcal{L}$, then there is some $A \in \mathcal{L}$ such that $x \in A$ and $A \cap B = \emptyset$ (0.31); so $A \in \varphi_{\mathcal{L}}(x)$ and no filter containing A can contain B.

Remark 0.32

Since $\mathcal{L}$ is a base for the closed sets in X, it clearly is a distributive lattice with unit X and zero element $\emptyset$.

Corollary 0.33

$\varphi_{\mathcal{L}}$ is a function from X to $\mathcal{W}(\mathcal{L})$.

Proposition 0.34

For any $x \in X$ and any $A \in \mathcal{L}$, $\varphi_{\mathcal{L}}(x) \in C(A)$ if and only if $x \in A$.

Proof: If $x \in A$, then, by Definition 0.31, $A \in \varphi_{\mathcal{L}}(x)$; so by Proposition 0.32 and Definition 0.24, $\varphi_{\mathcal{L}}(x) \in C(A)$. If $x \notin A$, then from the fact that $\mathcal{L}$ satisfies the conditions of Definition 0.31, there is some $B \in \mathcal{L}$ such that $x \in B$ and $A \cap B = \emptyset$. Hence, $\varphi_{\mathcal{L}}(x) \in C(B)$ (0.31) and $C(A)$ and $C(B)$ are disjoint (0.26 and 0.27). Thus $\varphi_{\mathcal{L}}(x) \notin C(A)$.

Proposition 0.35

For any $A \in \mathcal{L}$, $\overline{\varphi_{\mathcal{L}}[A]} = C(A)$.

Proof: Clearly for any $x \in A$, $A \in \varphi_{\mathcal{L}}(x)$; so $\varphi_{\mathcal{L}}(x) \in C(A)$. Since $C(A)$ is a closed set containing $\varphi_{\mathcal{L}}[A]$, $\overline{\varphi_{\mathcal{L}}[A]} \subseteq C(A)$. If $u \in \mathcal{W}(\mathcal{L}) \sim \overline{\varphi_{\mathcal{L}}[A]}$, then since $\{u\}$ is a closed subset of $\mathcal{W}(\mathcal{L})$ and $\{C(M): M \in \mathcal{L}\}$ is a base for the closed sets of $\mathcal{W}(\mathcal{L})$, there is some subcollection $\{M_\alpha : \alpha \in \mathcal{A}\} \subseteq \mathcal{L}$ such that $\{u\} = \bigcap_{\alpha \in \mathcal{A}} C(M_\alpha)$. Then $\{\mathcal{W}(\mathcal{L}) \sim C(M_\alpha) : \alpha \in \mathcal{A}\}$ is an open cover of the compact set $\overline{\varphi_{\mathcal{L}}[A]}$; so it must contain a finite subcover $\{\mathcal{W}(\mathcal{L}) \sim C(M_{\alpha_i}) : i=1,2,,n\}$. Then $\{u\} = \bigcap_{\alpha \in \mathcal{A}} C(M_\alpha) \subseteq \bigcap_{i=1}^{n} C(M_{\alpha_i})$. By Proposition 0.26, $\bigcap_{i=1}^{n} C(M_{\alpha_i}) = C(\bigcap_{i=1}^{n} M_{\alpha_i})$. Since $\{\mathcal{W}(\mathcal{L}) \sim C(M_{\alpha_i}) : i=1,2,,n\}$ covers $\overline{\varphi_{\mathcal{L}}[A]}$, $\overline{\varphi_{\mathcal{L}}[A]} \subseteq \bigcup_{i=1}^{n} (\mathcal{W}(\mathcal{L}) \sim C(M_{\alpha_i})) = \mathcal{W}(\mathcal{L}) \sim \bigcap_{i=1}^{n} C(M_{\alpha_i}) = \mathcal{W}(\mathcal{L}) \sim C(\bigcap_{i=1}^{n} M_{\alpha_i})$. Hence, $C(\bigcap_{i=1}^{n} M_{\alpha_i}) \cap \overline{\varphi_{\mathcal{L}}[A]} = \emptyset$; so $A \cap (\bigcap_{i=1}^{n} M_{\alpha_i}) = \emptyset$ (0.34). Therefore, from Propositions 0.26 and 0.27, $C(A) \cap C(\bigcap_{i=1}^{n} M_{\alpha_i}) = C(A \cap (\bigcap_{i=1}^{n} M_{\alpha_i})) = \emptyset$; so $u \notin C(A)$. Hence $u \in \overline{\varphi_{\mathcal{L}}[A]}$ if and only if $u \in C(A)$.

Proposition 0.36

$\varphi_{\mathcal{L}}$ is a dense embedding.

Proof: That $\varphi_{\mathcal{L}}$ is relatively open follows immediately from Proposition 0.34. $\varphi_{\mathcal{L}}$ is continuous since given any closed set A in

$\mathcal{W}(\mathcal{L})$, A is an intersection $\bigcap_{\alpha \in \mathcal{a}} C(B_\alpha)$ for some subcollection $\{B_\alpha : \alpha \in \mathcal{a}\}$ of $\mathcal{L}$ (since $\{C(B) : B \in \mathcal{L}\}$ is a base for the closed sets of $\mathcal{W}(\mathcal{L})$). Then $\varphi_{\mathcal{L}}^{-1}[A] = \varphi_{\mathcal{L}}^{-1}[\bigcap_{\alpha \in \mathcal{a}} C(B)] = \bigcap_{\alpha \in \mathcal{a}} \varphi_{\mathcal{L}}^{-1}[C(B_\alpha)]$. This is closed since, from Proposition 0.34, $\varphi_{\mathcal{L}}^{-1}[C(B_\alpha)] = B_\alpha$. Finally, $\varphi_{\mathcal{L}}$ is dense since, for any nonempty open set U in $\mathcal{W}(\mathcal{L})$, $\mathcal{W}(\mathcal{L}) \sim U$ is closed and therefore is the intersection $\bigcap_{\alpha \in \mathcal{a}} C(B_\alpha)$ for some subcollection $\{B_\alpha : \alpha \in \mathcal{a}\}$ of $\mathcal{L}$. Since $\bigcap_{\alpha \in \mathcal{a}} C(B_\alpha)$ is a proper subset of $\mathcal{W}(\mathcal{L})$, at least one $C(B_\beta)$ is a proper subset of $\mathcal{W}(\mathcal{L})$, which implies that B_β is a proper subset of X, since $x \in u$ for every $u \in \mathcal{W}(\mathcal{L})$. Hence, there is some $x \in X \sim B_\beta$. Thus from Proposition 0.35, $\varphi_{\mathcal{L}}(x) \in \mathcal{W}(\mathcal{L}) \sim C(B_\beta) \subseteq \mathcal{W}(\mathcal{L}) \sim \bigcap_{\alpha \in \mathcal{a}} C(B_\alpha) = U$.

<u>Corollary 0.37</u>

If X is compact, then $\varphi_{\mathcal{L}}$ is a homeomorphism.

<u>Proof</u>: If X is compact, each ultrafilter u in $\mathcal{L}$ must have nonempty intersection. If $x \in \bigcap u$, then, by Definition 0.31, Proposition 0.32, and the maximality of ultrafilters, $u = \varphi_{\mathcal{L}}(x)$; hence, $\varphi_{\mathcal{L}}$ carries X onto $\mathcal{W}(\mathcal{L})$ and is, by Proposition 0.36, continuous, one-to-one, and relatively open. Thus, $\varphi_{\mathcal{L}}$ is a homeomorphism.

<u>Corollary 0.38</u>

$\langle \mathcal{W}(\mathcal{L}), \varphi_{\mathcal{L}} \rangle$ is an extension of X.

We now consider the preservation of certain mapping properties under products.

For the remainder of this chapter we will assume that Λ is a set, for each $\lambda \in \Lambda$, X_λ and Y_λ are spaces, and f_λ is a function from X_λ to Y_λ.

Proposition 0.39

If f_λ is one-to-one for each $\lambda \in \Lambda$, then $\prod_{\lambda \in \Lambda} f_\lambda$ is one-to-one.

Proof: If $(x_\lambda)_{\lambda \in \Lambda}$ and $(z_\lambda)_{\lambda \in \Lambda}$ are distinct points of $\prod_{\lambda \in \Lambda} X_\lambda$, then there is some $r \in \Lambda$ such that $x_r \neq z_r$. Since f_r is one-to-one, $f_r(x_r) \neq f_r(z_r)$; hence $(\prod_{\lambda \in \Lambda} f_\lambda)((x_\lambda)_{\lambda \in \Lambda}) = (f_\lambda(x_\lambda))_{\lambda \in \Lambda} \neq (f_\lambda(z_\lambda))_{\lambda \in \Lambda} = (\prod_{\lambda \in \Lambda} f_\lambda)((z_\lambda)_{\lambda \in \Lambda})$.

Proposition 0.40

If f_λ is continuous for each $\lambda \in \Lambda$, then $\prod_{\lambda \in \Lambda} f_\lambda$ is continuous.

Proof: If $(x_\lambda)_{\lambda \in \Lambda}$ is a point of $\prod_{\lambda \in \Lambda} X_\lambda$ and if U is an open subset of $\prod_{\lambda \in \Lambda} Y_\lambda$ containing $(\prod_{\lambda \in \Lambda} f_\lambda)((x_\lambda)_{\lambda \in \Lambda})$, then there is some finite subset A of Λ and some collection $\{U_\alpha \subseteq Y_\alpha : \alpha \in A\}$ of open sets such that $(\prod_{\lambda \in \Lambda} f_\lambda)((x_\lambda)_{\lambda \in \Lambda}) \in \bigcap_{\alpha \in A} \pi_\alpha^{-1}(U_\alpha) \subseteq U$. Since each f_λ is continuous, then for each $\alpha \in A$, $f_\alpha^{-1}[U_\alpha]$ is an open subset of X_α containing x_α. Hence, $(x_\lambda)_{\lambda \in \Lambda} \in \bigcap_{\alpha \in A} \pi_\alpha^{-1}[f_\alpha^{-1}[U_\alpha]]$, and for any $(z_\lambda)_{\lambda \in \Lambda}$ in $\bigcap_{\alpha \in A} \pi_\alpha^{-1}[f_\alpha^{-1}[U_\alpha]]$, $f_\alpha(z_\alpha) \in U_\alpha$ for each $\alpha \in A$; so $(\prod_{\lambda \in \Lambda} f_\lambda)((z_\lambda)_{\lambda \in \Lambda}) \in \bigcap_{\alpha \in A} \pi_\alpha^{-1}[U_\alpha] \subseteq U$. Hence, $\prod_{\lambda \in \Lambda} f_\lambda$ is continuous.

Proposition 0.41

If f_λ is relatively open for each $\lambda \in \Lambda$, then $\prod_{\lambda \in \Lambda} f_\lambda$ is relatively open.

Proof: If $v = (x_\lambda)_{\lambda \in \Lambda}$ is an element of an open subset U of $\prod_{\lambda \in \Lambda} X_\lambda$, then there is some finite subset A_v of Λ and a collection of open sets $\{U_{v_\alpha} \subseteq X_\alpha : \alpha \in A_v\}$ such that $(x_\lambda)_{\lambda \in \Lambda} \in \bigcap_{\alpha \in A_v} \pi_\alpha^{-1}[U_{v_\alpha}] \subseteq U$. Since f_λ is relatively open for

each $\lambda \in \Lambda$, we have that for each $\alpha \in A_V$ there is some open $V_{V_\alpha} \subseteq Y_\alpha$ such that $f_\alpha[U_{V_\alpha}] = f_\alpha[X_\alpha] \cap V_{V_\alpha}$. Then $\bigcap_{\alpha \in A_V} \pi_\alpha^{-1}[V_{V_\alpha}]$ is an open subset of $\prod_{\lambda \in \Lambda} Y_\lambda$. For each $\alpha \in A$, $x_\alpha \in U_{V_\alpha}$; so $f_\alpha(x_\alpha) \in V_{V_\alpha}$. Hence, $(\prod_{\lambda \in \Lambda} f_\lambda)((x_\lambda)_{\lambda \in \Lambda}) = (f_\lambda(x_\lambda))_{\lambda \in \Lambda} \in \bigcap_{\alpha \in A_V} \pi_\alpha^{-1}[V_{V_\alpha}]$. Furthermore, if $(z_\lambda)_{\lambda \in \Lambda}$ is an element of $\prod_{\lambda \in \Lambda} X_\lambda$ and if $(\prod_{\lambda \in \Lambda} f_\lambda)((z_\lambda)_{\lambda \in \Lambda}) \notin \bigcap_{\alpha \in A_V} \pi_\alpha^{-1}[V_{V_\alpha}]$, then there is some $\beta \in A_V$ such that $f_\beta(z_\beta) \notin V_{V_\beta}$, which implies that $z \notin U_{V_\beta}$; so $(z_\lambda)_{\lambda \in \Lambda} \notin \bigcap_{\alpha \in A_V} \pi_\alpha^{-1}[U_{V_\alpha}]$. Therefore, $(\prod_{\lambda \in \Lambda} f_\lambda)[\bigcap_{\alpha \in A_V} \pi_\alpha^{-1}[U_{V_\alpha}]] = (\prod_{\lambda \in \Lambda} f_\lambda)[\prod_{\lambda \in \Lambda} X_\lambda] \cap (\bigcap_{\alpha \in A_V} \pi_\alpha^{-1}[V_{V_\alpha}])$; so $(\prod_{\lambda \in \Lambda} f_\lambda)[U] = (\prod_{\lambda \in \Lambda} f_\lambda)[\prod_{\lambda \in \Lambda} X_\lambda] \cap (\bigcup_{V \in U}(\bigcap_{\alpha \in A_V} \pi_\alpha^{-1}[V_{V_\alpha}]))$.

Proposition 0.42

If f_λ is dense for each $\lambda \in \Lambda$, then $\prod_{\lambda \in \Lambda} f_\lambda$ is dense.

Proof: Let U be a nonempty open subset of $\prod_{\lambda \in \Lambda} Y_\lambda$. Then there is some finite subset A of Λ and a collection $\{U_\alpha \subseteq Y_\alpha : \alpha \in A\}$ of nonempty open sets such that $\bigcap_{\alpha \in A} \pi_\alpha^{-1}[U_\alpha] \subseteq U$. For each $\lambda \in \Lambda$, define $V_\lambda = \begin{cases} Y_\lambda & \text{if } \lambda \notin A \\ U_\lambda & \text{if } \lambda = \alpha \in A. \end{cases}$ Then for each $\lambda \in \Lambda$, $f_\lambda^{-1}[V_\lambda]$ is a nonempty subset of X_λ; so $\prod_{\lambda \in \Lambda} f_\lambda^{-1}[V_\lambda]$ is a nonempty subset of $\prod_{\lambda \in \Lambda} X_\lambda$. If $(x_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} f_\lambda^{-1}[V_\lambda]$, then for each $\lambda \in \Lambda$, $f_\lambda(x_\lambda) \in V_\lambda$; so $(\prod_{\lambda \in \Lambda} f_\lambda)((x_\lambda)_{\lambda \in \Lambda}) = (f_\lambda(x_\lambda))_{\lambda \in \Lambda} \in \bigcap_{\alpha \in A} \pi_\alpha^{-1}[U_\alpha] \subseteq U$.

Corollary 0.43

If $\langle Y_\lambda, f_\lambda \rangle$ is an extension of X_λ for each $\lambda \in \Lambda$, then $\langle \prod_{\lambda \in \Lambda} Y_\lambda, \prod_{\lambda \in \Lambda} f_\lambda \rangle$ is an extension of $\prod_{\lambda \in \Lambda} X_\lambda$.

Proof: This is immediate from Propositions 0.40, 0.41, and 0.42.

Proposition 0.44

If f_λ is continuous and dense for each $\lambda \in \Lambda$, then for any open set $U \subseteq \prod_{\lambda \in \Lambda} Y_\lambda$ and any $\alpha \in \Lambda$, $f_\alpha^{-1}[\pi_\alpha[U]] = \pi_\alpha[(\prod_{\lambda \in \Lambda} f_\lambda)^{-1}[U]]$.

Proof: If $x_\alpha \in \pi_\alpha[(\prod_{\lambda \in \Lambda} f_\lambda)^{-1}[U]]$, then there is some point $(w_\lambda)_{\lambda \in \Lambda}$ of $\prod_{\lambda \in \Lambda} X_\lambda$ such that $w_\alpha = x_\alpha$ and $(\prod_{\lambda \in \Lambda} f_\lambda)((w_\lambda)_{\lambda \in \Lambda}) = (f_\lambda(w_\lambda))_{\lambda \in \Lambda} \in U$, which implies $f_\alpha(w_\alpha) = f_\alpha(x_\alpha) \in \pi_\alpha[U]$; so $x_\alpha \in f_\alpha^{-1}[\pi_\alpha[U]]$. On the other hand, if x_α is an element of $f_\alpha^{-1}[\pi_\alpha[U]]$, then $f_\alpha(x_\alpha) \in \pi_\alpha[U]$; so there is some element $(w_\lambda)_{\lambda \in \Lambda}$ of U such that $w_\alpha = f(x_\alpha)$. Then for each $\lambda \in \Lambda$, there is some open $U_\lambda \subseteq Y_\lambda$ such that $w_\lambda \in U_\lambda$ and $\prod_{\lambda \in \Lambda} U_\lambda \subseteq U$. For each $\lambda \in \Lambda$, $f_\lambda^{-1}[U_\lambda]$ is nonempty; so we can choose $z_\lambda \in f_\lambda^{-1}[U_\lambda]$ with the provision that $z_\alpha = x_\alpha$. Then $(z_\lambda)_{\lambda \in \Lambda} \in (\prod_{\lambda \in \Lambda} f_\lambda)^{-1}[U]$ and $x_\alpha = z_\alpha = \pi_\alpha((z_\lambda)_{\lambda \in \Lambda}) \in \pi_\alpha[(\prod_{\lambda \in \Lambda} f_\lambda)^{-1}[U]]$.

CHAPTER 1

EXTENSIONS

Definition 1.1

An extension $\langle Y, \varphi \rangle$ of a space X is called distinguishable provided that for any two distinct points of Y there is an open set U in Y containing one of them such that $\varphi^{-1}[U] \not\subset \varphi^{-1}[V]$ for any open set V containing the other.

Definition 1.2

An extension $\langle Y, \varphi \rangle$ of a space X is called bidistinguishable provided that for any two distinct points y and z of Y there is some open set U containing y such that $\varphi^{-1}[U] \not\subset \varphi^{-1}[V]$ for any open set V containing z.

Definition 1.3

An extension $\langle Y, \varphi \rangle$ of a space X is called C-distinguishable provided that $\langle Y, \varphi \rangle$ is distinguishable and for any point y of Y and any closed set A not containing y there is some open set U containing y such that $\varphi^{-1}[U] \not\subset \varphi^{-1}[V]$ for any open set V such that $V \cap A \neq \emptyset$.

Definition 1.4

An extension $\langle Y, \varphi \rangle$ of a space X will be called relatively Hausdorff provided that for any point x of X and any point y of Y

distinct from $\varphi(x)$, there exist disjoint open sets U and V in Y such that $y \in U$ and $\varphi(x) \in V$.

Proposition 1.5

If $\langle Y, \varphi\rangle$ is an extension of a space X and if Y is a Hausdorff space, then $\langle Y, \varphi\rangle$ is bidistinguishable and relatively Hausdorff.

Proof: Since Y is Hausdorff, $\langle Y, \varphi\rangle$ is relatively Hausdorff. If y and z are distinct elements of Y and if U and V are disjoint open sets containing y and z, respectively, then $\varphi^{-1}[U]$ and $\varphi^{-1}[V]$ are disjoint open subsets of X. If W is any open set containing z, then $W \cap V$ is a nonempty open subset of Y; so that since φ is dense, $\emptyset \neq \varphi^{-1}[W \cap V] \subseteq \varphi^{-1}[W]$. But $\varphi^{-1}[W \cap V] \subseteq \varphi^{-1}[V]$ and $\varphi^{-1}[V]$ is disjoint from $\varphi^{-1}[U]$ so $\varphi^{-1}[W] \not\subseteq \varphi^{-1}[U]$. Hence, $\langle Y, \varphi\rangle$ is bidistinguishable.

For an example of a Hausdorff extension which is not C-distinguishable, consider the space X consisting of the closed interval $[0,1]$ with topology generated by the usual open sets in $[0,1]$ and the set Q of rational elements of $[0,1]$. This is a Hausdorff space and an extension of the subspace Q' consisting of the rational numbers in $(0,1)$. It is not C-distinguishable because the irrationals in $[0,1]$ form a closed set, but the intersection of any open set in X with Q' is the same as the intersection with Q' of an open set which contains some irrational numbers.

Proposition 1.6

If $\langle Y, \varphi\rangle$ is an extension of X and if Y is a T_3 space, then $\langle Y, \varphi\rangle$ is C-distinguishable.

Proof: From Proposition 1.5, $<Y, \varphi>$ is distinguishable. If A is a closed subset of Y and if $y \in Y \sim A$, then there exist disjoint open sets U and V in Y such that $y \in U$ and $A \subseteq V$. Since U and V are disjoint, $\varphi^{-1}[U]$ and $\varphi^{-1}[V]$ are disjoint. If W is an open subset of Y containing any point of A, then $W \cap V$ is a non-empty open subset of Y so, since φ is dense, $\emptyset \neq \varphi^{-1}[W \cap V] \subseteq \varphi^{-1}[W]$. But $\varphi^{-1}[W \cap V] \subseteq \varphi^{-1}[V]$; so $\varphi^{-1}[W] \nsubseteq \varphi^{-1}[U]$.

Corollary 1.7

A Hausdorff compactification is a C-distinguishable bidistinguishable, relatively Hausdorff extension.

Proposition 1.8

Let X be a T_1 space and let $\mathcal{L}$ be a lattice of closed subsets of X which is a base for the closed sets in X and such that for any point $x \in X$ and any element $A \in \mathcal{L}$ which does not contain x there exists some $B \in \mathcal{L}$ with $x \in B$ and $A \cap B = \emptyset$. Then $< \mathcal{W}(\mathcal{L}), \varphi_{\mathcal{L}} >$ is a bidistinguishable C-distinguishable extension of X.

Proof: From Corollary 0.38 $< \mathcal{W}(\mathcal{L}), \varphi_{\mathcal{L}} >$ is an extension of X. If u and v are distinct elements of $\mathcal{W}(\mathcal{L})$, then by Proposition 0.15 there exist disjoint sets A and B in $\mathcal{L}$ such that $A \in u$ and $B \in v$. From Definition 0.24, Proposition 0.26, and Proposition 0.27, C(B) is a closed subset of $\mathcal{W}(\mathcal{L})$ containing v but not u. Hence $\mathcal{W}(\mathcal{L}) \sim C(B)$ is an open subset of $\mathcal{W}(\mathcal{L})$ which contains u. From Proposition 0.34, $\varphi_{\mathcal{L}}^{-1}[\mathcal{W}(\mathcal{L}) \sim C(B)] = X \sim B$. If U is an open subset of $\mathcal{W}(\mathcal{L})$ which contains v, then $\mathcal{W}(\mathcal{L}) \sim U$ is a closed subset of $\mathcal{W}(\mathcal{L})$ which does not contain v. The collection $\{C(M): M \in \mathcal{L}\}$ is a base for the closed sets in $\mathcal{W}(\mathcal{L})$; so $\mathcal{W}(\mathcal{L}) \sim U = \bigcap_{\alpha \in \mathcal{a}} C(M_\alpha)$ for some subcollection $\{M_\alpha : \alpha \in \mathcal{a}\}$ of $\mathcal{L}$. Since $v \notin \bigcap_{\alpha \in \mathcal{a}} C(M_\alpha)$,

there is some $\beta \in \mathcal{a}$ such that $v \notin C(M_\beta)$. By Definition 0.24, $M_\beta \notin v$; so by Proposition 0.16 there is some $E \in v$ such that $E \cap M_\beta = \emptyset$; hence, $E \subseteq X \sim M_\beta$. Then from Proposition 0.34,

$$E \subseteq X \sim M_\beta = \varphi_{\mathcal{L}}^{-1}[\mathcal{W}(\mathcal{L}) \sim C(M_\beta)] \subseteq \varphi_{\mathcal{L}}^{-1}[\mathcal{W}(\mathcal{L}) \sim \bigcap_{\alpha \in \mathcal{a}} C(M_\alpha)].$$

But $E \cap B \neq \emptyset$; so $E \nsubseteq X \sim B$; hence

$$\varphi_{\mathcal{L}}^{-1}[U] = \varphi_{\mathcal{L}}^{-1}[\mathcal{W}(\mathcal{L}) \sim \bigcap_{\alpha \in \mathcal{a}} C(M_\alpha)] \nsubseteq X \sim B = \varphi_{\mathcal{L}}^{-1}[\mathcal{W}(\mathcal{L}) \sim C(B)].$$

Thus, $<\mathcal{W}(\mathcal{L}), \varphi_{\mathcal{L}}>$ is bidistinguishable. If A is a closed subset of $\mathcal{W}(\mathcal{L})$ and if $u \in \mathcal{W}(\mathcal{L}) \sim A$, then since $\{C(M): M \in \mathcal{L}\}$ is a base for the closed subsets of $\mathcal{W}(\mathcal{L})$, $A = \bigcap_{\alpha \in \mathcal{a}} C(M_\alpha)$ for some subcollection $\{M_\alpha : \alpha \in \mathcal{a}\}$ of $\mathcal{L}$. Hence, there is some $\beta \in \mathcal{a}$ such that $u \notin C(M_\beta)$. Then $\mathcal{W}(\mathcal{L}) \sim C(M_\beta)$ is an open set containing u and, as was just shown, for any $v \in A \subseteq C(M_\beta)$ and any open set U containing v, $\varphi_{\mathcal{L}}^{-1}[U] \nsubseteq X \sim M_\beta = \varphi_{\mathcal{L}}^{-1}[\mathcal{W}(\mathcal{L}) \sim C(M_\beta)]$. Hence, $<\mathcal{W}(\mathcal{L}), \varphi_{\mathcal{L}}>$ is C-distinguishable.

<u>Proposition 1.9</u>

If X is a T_3 space, then $<\mathcal{W}(\mathcal{L}_X), \varphi_{\mathcal{L}_X}>$ is a relatively Hausdorff extension of X.

<u>Proof</u>: Clearly $\mathcal{L}_X$ satisfies the standing hypotheses used in Corollary 0.38. Thus $<\mathcal{W}(\mathcal{L}_X), \varphi_{\mathcal{L}_X}>$ is an extension of X. Let u and $\varphi_{\mathcal{L}_X}(x)$ be distinct elements of $\mathcal{W}(\mathcal{L}_X)$. By Proposition 0.15 there exist disjoint A and B in $\mathcal{L}_X$ such that $A \in \varphi_{\mathcal{L}_X}(x)$ and $B \in u$. By Definition 0.31, $x \in A$; so $x \notin B$ and B is closed in X; hence, there exist disjoint open sets U and V in X such that $x \in V$ and $B \subseteq U$. Then $X \sim U$ and $X \sim V$ are elements of $\mathcal{L}_X$ and $(X \sim U) \cup (X \sim V) = X$. $(X \sim V) \in u$, since $B \subseteq U \subseteq X \sim V$ and $B \in u$. $(X \sim U) \in \varphi_{\mathcal{L}_X}(x)$, since $x \in (X \sim U) \in \mathcal{L}_X$. Furthermore, since $B \cap (X \sim U) = \emptyset$, it follows that $(X \sim U) \notin u$; so $u \notin C(X \sim U)$ and

$\varphi_{\mathcal{L}_X}(x) \notin C(X \sim V)$. By Proposition 0.25, $\mathcal{W}(\mathcal{L}_X) = C(X) = C((X \sim U) \cup (X \sim V)) = C(X \sim U) \cup C(X \sim V)$. Therefore, $\mathcal{W}(\mathcal{L}_X) \sim C(X \sim U)$ and $\mathcal{W}(\mathcal{L}_X) \sim C(X \sim V)$ are disjoint open sets in $\mathcal{W}(\mathcal{L}_X)$ containing u and $\varphi_{\mathcal{L}_X}(x)$, respectively.

Theorem 1.10

A space X is T_3 if and only if it has a compact relatively Hausdorff extension.

Proof: If X is T_3, then $\langle \mathcal{W}(\mathcal{L}_X), \varphi_{\mathcal{L}_X} \rangle$ is a relatively Hausdorff extension of X by Proposition 1.9 and $\mathcal{W}(\mathcal{L}_X)$ is compact by Proposition 0.29. Conversely, let $\langle Y, \varphi \rangle$ be a compact relatively Hausdorff extension of X. Since φ is an embedding, for any closed set A in X and any point x not contained in A there is some closed set B in Y such that $\varphi[A] = \varphi[X] \cap B$. Since φ is one-to-one, $\varphi(x) \notin \varphi[A]$; so $\varphi(x) \notin B$; hence for each $y \in B$ there exist disjoint open sets U_y and V_y in Y such that $\varphi(x) \in U_y$ and $y \in V_y$. Since B is a closed subset of a compact space Y and since $\{V_y : y \in B\}$ is an open cover of B, there is a finite subcover $\{V_{y_i} : i=1,2,,n\}$. Then $\bigcup_{i=1}^{n} V_{y_i}$ and $\bigcap_{i=1}^{n} U_{y_i}$ are disjoint open sets in Y containing B and $\varphi(x)$, respectively. Hence $\varphi^{-1}[\bigcup_{i=1}^{n} V_{y_i}]$ and $\varphi^{-1}[\bigcap_{i=1}^{n} U_{y_i}]$ are disjoint open sets in X containing A and x, respectively.

Proposition 1.11

If Y is a T_1 space and if $\langle Y, \varphi \rangle$ is a C-distinguishable extension of X, then $\langle Y, \varphi \rangle$ is bidistinguishable.

Proof: If y and z are distinct points of Y, then $\{z\}$ is a closed set since Y is T_1; hence, since $\langle Y, \varphi \rangle$ is C-distinguishable, there is some open subset U of Y containing y such that $\varphi^{-1}[U] \not\subset \varphi^{-1}[V]$ for any open set V containing z.

Proposition 1.12

If $\varphi[X] \subseteq Y \subseteq Z$ and if $<Z, \varphi>$ is a distinguishable, bidistinguishable, or C-distinguishable extension of X, then so is $<Y, \varphi>$.

Proof: Given any open set U in Y, then $U = Y \cap V$ for some open set V in Z. Since $\varphi[X]$ is dense in Z, there is some point $x \in X$ such that $\varphi(x) \in V$. Then because $\varphi[X] \subseteq Y$, $\varphi(x) \in Y \cap V = U$; hence, $\varphi[X]$ is dense in Y; so $<Y, \varphi>$ is an extension of X. If $<Z, \varphi>$ is distinguishable and if y and z are distinct points of Y, then y and z are distinct points of Z; so there is an open set U in Z containing one (say, y) such that $\varphi^{-1}[U] \not\subseteq \varphi^{-1}[V]$ for any open set V containing the other. Then $y \in U \cap Y$, $U \cap Y$ is open in Y, and $\varphi^{-1}[U \cap Y] = \varphi^{-1}[U]$. If $z \in W$ for some open set W of Y, then $W = Y \cap V$ for some open set V in Z and $z \in V$; so $\varphi^{-1}[W] = \varphi^{-1}[V] \not\supseteq \varphi^{-1}[U] = \varphi^{-1}[U \cap Y]$; hence, $<Y, \varphi>$ is distinguishable. If $<Z, \varphi>$ is bidistinguishable, the same argument with the phrase "containing one (say, y)" replaced by the phrase "containing y" proves that $<Y, \varphi>$ is bidistinguishable. If $<Z, \varphi>$ is C-distinguishable, then $<Z, \varphi>$ is distinguishable; so $<Y, \varphi>$ is distinguishable. If $y \in U$ for some open set U in Y, then there is an open set W in Z such that $W \cap Y = U$. Now since $<Z, \varphi>$ is C-distinguishable, there is an open set V in Z containing y such that $\varphi^{-1}[V] \not\subseteq \varphi^{-1}[V']$ for any open set V' containing any point of $Z \sim W$. If V_0 is an open set in Y containing any point z of $Y \sim U$, then there is some open set V_0' in Z such that $V_0' \cap Y = V_0$. Since $z \in V_0$, $z \in V_0'$. Since $z \notin U$, $z \notin W$. Hence, $\varphi^{-1}[V \cap Y] = \varphi^{-1}[V] \not\subseteq \varphi^{-1}[V_0'] = \varphi^{-1}[V_0]$. Therefore, $<Y, \varphi>$ is C-distinguishable.

Proposition 1.13

If $X \subseteq Y$ and if $\langle Z, \varphi\rangle$ is a relatively Hausdorff extension of Y, then $\langle \overline{\varphi[X]}, \varphi/X\rangle$ is a relatively Hausdorff extension of X.

Proof: It is clear that $\varphi[X]$ is dense in $\overline{\varphi[X]}$ and that φ, being a homeomorphism of Y onto $\varphi[Y]$, implies that φ/X is a homeomorphism of X onto $\varphi[X]$. Hence $\langle \overline{\varphi[X]}, \varphi/X\rangle$ is an extension of X. Given distinct points y and $\varphi(x)$ in $\overline{\varphi(X)}$, then since y and $\varphi(x)$ are distinct points in Z and $\langle Z, \varphi\rangle$ is relatively Hausdorff, there exist disjoint open sets U and V in Z containing y and $\varphi(x)$, respectively. Hence, $U \cap \overline{\varphi(X)}$ and $V \cap \overline{\varphi(X)}$ are disjoint open sets in $\overline{\varphi(X)}$ containing y and $\varphi(x)$, respectively.

Proposition 1.14

If $\langle Y, \varphi\rangle$ is an extension of X and if $\langle Z, \psi\rangle$ is a relatively Hausdorff extension of Y, then $\langle Z, \psi \circ \varphi\rangle$ is a relatively Hausdorff extension of X.

Proof: It is well known that the composition of dense embeddings is a dense embedding. Thus, $\langle Z, \psi \circ \varphi\rangle$ is an extension of X. If z and $\psi \circ \varphi(x)$ are distinct points of Z, then since $\langle Z, \psi\rangle$ is a relatively Hausdorff extension of Y and $\varphi(x) \in Y$, there exist disjoint open sets U and V in Z such that $z \in U$ and $\psi \circ \varphi(x) \in V$; hence, $\langle Z, \psi \circ \varphi\rangle$ is relatively Hausdorff.

Theorem 1.15

If for each element λ of a set Λ, $\langle Y_\lambda, \varphi_\lambda\rangle$ is an extension of X_λ which is relatively Hausdorff, C-distinguishable, bidistinguishable, or distinguishable, then the extension $\langle \prod_{\lambda \in \Lambda} Y_\lambda, \prod_{\lambda \in \Lambda} \varphi_\lambda\rangle$ of $\prod_{\lambda \in \Lambda} X_\lambda$ is, respectively, relatively Hausdorff, C-distinguishable, bidistinguishable, or distinguishable.

Proof:

(i) If each $<Y_\lambda, \varphi_\lambda>$ is a relatively Hausdorff extension of X_λ and if $(y_\lambda)_{\lambda \in \Lambda}$ and $(\varphi_\lambda(x_\lambda))_{\lambda \in \Lambda}$ are distinct points of $\prod_{\lambda \in \Lambda} Y_\lambda$, then there is some $\alpha \in \Lambda$ such that $\varphi_\alpha(x_\alpha) \neq y_\alpha$. Then since $<Y_\alpha, \varphi_\alpha>$ is a relatively Hausdorff extension of X_α, there exist disjoint open sets U and V of Y_α such that $y_\alpha \in U$ and $\varphi_\alpha(x_\alpha) \in V$. Hence, $\pi_\alpha^{-1}[U]$ and $\pi_\alpha^{-1}[V]$ are disjoint open subsets of $\prod_{\lambda \in \Lambda} Y_\lambda$ and $(y_\lambda)_{\lambda \in \Lambda} \in \pi_\alpha^{-1}[U]$ and $(\varphi_\lambda(x_\lambda))_{\lambda \in \Lambda} \in \pi_\alpha^{-1}[V]$.

(ii) If each $<Y_\lambda, \varphi_\lambda>$ is a distinguishable extension of X_λ and if $(y_\lambda)_{\lambda \in \Lambda}$ and $(z_\lambda)_{\lambda \in \Lambda}$ are distinct elements of $\prod_{\lambda \in \Lambda} Y_\lambda$, then there is some $\alpha \in \Lambda$ such that $y_\alpha \neq z_\alpha$. Because $<Y_\alpha, \varphi_\alpha>$ is distinguishable, there is some open U in Y_α containing one (say, y_α) such that $\varphi_\alpha^{-1}[U] \not\subseteq \varphi_\alpha^{-1}[V]$ for any open V in Y_α containing z_α. Then $\pi_\alpha^{-1}[U]$ is an open subset of $\prod_{\lambda \in \Lambda} Y_\lambda$ which contains $(y_\lambda)_{\lambda \in \Lambda}$. If W is an open subset of $\prod_{\lambda \in \Lambda} Y_\lambda$ containing $(z_\lambda)_{\lambda \in \Lambda}$, then $\pi_\alpha[W]$ is an open subset of Y_α containing z_α. Hence, from the above and Proposition 0.44,

$\pi_\alpha[(\prod_{\lambda \in \Lambda} \varphi_\lambda)^{-1}[W]] = \varphi_\alpha^{-1}[\pi_\alpha[W]] \not\subseteq \varphi_\alpha^{-1}[U] = \varphi_\alpha^{-1}[\pi_\alpha[\pi_\alpha^{-1}[U]]] =$ $\pi_\alpha[(\prod_{\lambda \in \Lambda} \varphi_\lambda)^{-1}[\pi_\alpha^{-1}[U]]]$ which implies $(\prod_{\lambda \in \Lambda} \varphi_\lambda)^{-1}[W] \not\subseteq (\prod_{\lambda \in \Lambda} \varphi_\alpha)^{-1}[\pi_\alpha^{-1}[U]]$; so $< \prod_{\lambda \in \Lambda} Y_\lambda, \prod_{\lambda \in \Lambda} \varphi_\lambda >$ is a distinguishable extension of $\prod_{\lambda \in \Lambda} X_\lambda$.

(iii) If each $<Y_\lambda, \varphi_\lambda>$ is bidistinguishable, the same proof except for the substitution of the phrase "containing y_α" for "containing one (say, y_α)" shows that $< \prod_{\lambda \in \Lambda} Y_\lambda, \prod_{\lambda \in \Lambda} \varphi_\lambda >$ is a bidistinguishable extension of $\prod_{\lambda \in \Lambda} X_\lambda$.

(iv) If each $<Y_\lambda, \varphi_\lambda>$ is C-distinguishable, then from (ii) above, $< \prod_{\lambda \in \Lambda} Y_\lambda, \prod_{\lambda \in \Lambda} \varphi_\lambda >$ is a distinguishable extension of $\prod_{\lambda \in \Lambda} X_\lambda$.

Given an element $(y_\lambda)_{\lambda\in\Lambda}$ of an open subset U of $\prod_{\lambda\in\Lambda} Y_\lambda$, there exists some finite collection $\{U_{\lambda_i} \subseteq Y_{\lambda_i}: i=1,2,,n\}$ of open sets such that $(y_\lambda)_{\lambda\in\Lambda} \in \bigcap_{i=1}^{n} \pi_{\lambda_i}^{-1}[U_{\lambda_i}] \subseteq U$. Since each $\langle Y_{\lambda_i}, \varphi_{\lambda_i}\rangle$ is C-distinguishable, for each λ_i there exists an open set $V_{\lambda_i} \subseteq U_{\lambda_i}$ such that $y_{\lambda_i} \in V_{\lambda_i}$ and for any open $W \subseteq Y_{\lambda_i}$, if $W \cap (Y_{\lambda_i} \sim U_{\lambda_i}) \neq \emptyset$, then $\varphi_{\lambda_i}^{-1}[W] \not\subseteq \varphi_{\lambda_i}^{-1}[V_{\lambda_i}]$. Then $(y_\lambda)_{\lambda\in\Lambda} \in \bigcap_{i=1}^{n} \pi_{\lambda_i}^{-1}[V_{\lambda_i}] \subseteq \bigcap_{i=1}^{n} \pi_{\lambda_i}^{-1}[U_{\lambda_i}] \subseteq U$. If $(z_\lambda)_{\lambda\in\Lambda} \notin U$, then $(z_\lambda)_{\lambda\in\Lambda} \notin \bigcap_{i=1}^{n} \pi_{\lambda_i}^{-1}[U_{\lambda_i}]$, which implies that there is some j for which $z_{\lambda_j} \notin U_{\lambda_j}$. If $(z_\lambda)_{\lambda\in\Lambda} \in W$ for some open subset W of $\prod_{\lambda\in\Lambda} Y_\lambda$, then $z_{\lambda_j} \in \pi_{\lambda_j}[W]$ and $\pi_{\lambda_j}[W]$ is an open subset of Y_{λ_j}; so from Proposition 0.44, $\pi_{\lambda_j}[(\prod_{\lambda\in\Lambda} \varphi_\lambda)^{-1}[W]] = \varphi_{\lambda_j}^{-1}[\pi_{\lambda_j}[W]] \not\subseteq \varphi_{\lambda_j}^{-1}[V_{\lambda_j}] = \varphi_{\lambda_j}^{-1}[\pi_{\lambda_j}[\pi_{\lambda_j}^{-1}[V_{\lambda_j}]]] = \pi_{\lambda_j}[(\prod_{\lambda\in\Lambda} \varphi_\lambda)[\bigcap_{i=1}^{n} \pi_{\lambda_i}^{-1}[V_{\lambda_i}]]]$, which implies $(\prod_{\lambda\in\Lambda} \varphi_\lambda)^{-1}[W] \not\subseteq (\prod_{\lambda\in\Lambda} \varphi_\lambda)^{-1}[\bigcap_{i=1}^{n} \pi_{\lambda_i}^{-1}[V_{\lambda_i}]]$. Hence, $\langle \prod_{\lambda\in\Lambda} Y_\lambda, \prod_{\lambda\in\Lambda} \varphi_\lambda\rangle$ is a C-distinguishable extension of $\prod_{\lambda\in\Lambda} X_\lambda$.

Proposition 1.16

If $\langle Y,\varphi\rangle$ is an extension of X, $\langle Z,\psi\rangle$ is an extension of W, and f is an embedding of Z into Y such that $\varphi[X] \subseteq f\circ\psi[W]$ (i.e., if there is a subset W' of W and a homeomorphism f' of W' onto X such that the diagram

$$\begin{tikzcd} W' \arrow[r, hook, "i"] \arrow[d, "f'"'] & W \arrow[r, "\psi"] & Z \arrow[d, "f"] \\ X \arrow[rr, "\varphi"'] & & Y \end{tikzcd}$$

commutes), then $\langle Z, \psi\rangle$ is C-distinguishable, bidistinguishable, or distinguishable if $\langle Y, \varphi\rangle$ is.

Proof: Let W' be $(f\circ\psi)^{-1}[\varphi[X]]$, and let i be the inclusion function from W' into W. Define f' carrying W' into X by $f'(z) = \bigcup \varphi^{-1}[f\circ\psi\circ i(z)]$. f' is well defined since φ is one-to-one, and (by the choice of W') for any $z \in W'$ there is at least one $x \in X$ such that $f\circ\psi(z) = \varphi(x)$. Thus $f'(z) =$ $\bigcup \varphi^{-1}[f\circ\psi\circ i(z)] \subseteq \bigcup \varphi^{-1}[f\circ\psi[(f\circ\psi)^{-1}[\varphi(x)]]] = \bigcup\{x\} = x$, since f, φ, and ψ are all one-to-one. If z_1 and z_2 are distinct elements of W', $\psi(z_1) \neq \psi(z_2)$, so $f\circ\psi(z_1) \neq f\circ\psi(z_2)$; hence, $f'(z_1) \neq f'(z_2)$, since $\varphi^{-1}[f\circ\psi(z_1)]$ and $\varphi^{-1}[f\circ\psi(z_2)]$ are disjoint. For any $x \in X$ there is by hypothesis some $w \in W$ such that $\varphi(x) = f\circ\psi(w)$. By the definition of W', $w \in W'$; and by definition of f' and the fact that it is well defined, $f'(w) = x$; hence, f' is onto. If U is an open subset of X, then, since φ is relatively open, there is some open set V in Y such that $\varphi[U] = V \cap \varphi[X]$. Since i, ψ, and f are continuous, $i^{-1}[\psi^{-1}[f^{-1}[V]]]$ is an open subset of W'. $w \in i^{-1}[\psi^{-1}[f^{-1}[V]]]$ implies that $f\circ\psi\circ i(w) \in V$ and (since $w \in W'$) that $f\circ\psi\circ i(w) = \varphi(x)$ for some $x \in X$; so $f\circ\psi\circ i(w) \in \varphi[X] \cap V$. Hence, $x = f'(w) \in U$; so $i^{-1}[\psi^{-1}[f^{-1}[V]]] = f'^{-1}[U]$. Therefore, f' is continuous. If U is an open subset of W', then $i[U] = W' \cap U_0$ where U_0 is open in W. Since ψ is one-to-one and relatively open, $\psi\circ i[U] = \psi[W' \cap U_0] =$ $\psi[W'] \cap \psi[U_0] = \psi[W'] \cap \psi[W] \cap U_1 = \psi[W'] \cap U_1$ (for some open set U_1 in Z). Because f is one-to-one and relatively open, there is an open set U_2 in Y such that $f\circ\psi\circ i[U] = f[\psi[W'] \cap U_1] =$ $f\circ\psi[W'] \cap f[U_1] = f\circ\psi[W'] \cap f[Z] \cap U_2 = f\circ\psi[W'] \cap U_2$. Then,

since φ is continuous, $\varphi^{-1}[f\circ\psi\circ i[U]] = \varphi^{-1}[f\circ\psi[W'] \cap U_2] = \varphi^{-1}[f\circ\psi[W']] \cap \varphi^{-1}[U_2] = X \cap \varphi^{-1}[U_2] = \varphi^{-1}[U_2]$, which is an open subset of X. Now $x \in \varphi^{-1}[U_2]$ implies $\varphi(x) = f\circ\psi\circ i(w)$ for some $w \in U$. Hence, $x = f'(w)$; so $\varphi^{-1}[U_2] = f'[U]$. Therefore, f' is one-to-one, onto, continuous, and open; i.e., a homeomorphism. Thus, if we let $\overline{f}$ denote the function f carrying X onto $f[Z]$, j the inclusion function of $f[Z]$ into Y, 1_X the identity function on X, and $\overline{\varphi}$ the function φ of X into $f[Z]$, then the following diagram commutes

$$\begin{tikzcd} W' \arrow[r, "f'"] \arrow[d, "i"'] & X \arrow[r, "1_X"] \arrow[dd, "\overline{\varphi}"] & X \arrow[dd, "\varphi"] \\ W \arrow[d, "\psi"'] & & \\ Z \arrow[r, "\overline{f}"'] & f[Z] \arrow[r, "j"'] & Y \end{tikzcd}$$

and $\overline{f}$ and f' are homeomorphisms.

Then from Proposition 1.12, if $\langle Y, \varphi \rangle$ is distinguishable, bidistinguishable, or C-distinguishable, so is $\langle f[Z], \overline{\varphi} \rangle$; and hence so is $\langle Z, \psi\circ i \rangle$. If $\langle Z, \psi\circ i \rangle$ is a distinguishable extension of W' and if y and z are distinct elements of Z, then there is some open set U in Z containing one (say, y) such that $(\psi\circ i)^{-1}[U] \not\subseteq (\psi\circ i)^{-1}[V]$ for any open set V containing z. But $(\psi\circ i)^{-1}[U] = i^{-1}[\psi^{-1}[U]]$ and $(\psi\circ i)^{-1}[V] = i^{-1}[\psi^{-1}[V]]$; so $\psi^{-1}[U] \not\subseteq \psi^{-1}[V]$. If the phrase "containing one (say, y)" is replaced by "containing y," the above proves the corresponding result for bidistinguishability. If $\langle Z, \psi\circ i \rangle$ is C-distinguishable, then for any $z \in Z$ and any open subset U of Z containing z, there is some open subset V of Z

containing z such that $(\psi \circ i)^{-1}[V] \not\subset (\psi \circ i)^{-1}[V']$ for any open set V' in Z such that $V' \cap (Z \sim U) \neq \emptyset$. But $i^{-1}[\psi^{-1}[V]] = (\psi \circ i)^{-1}[V] \not\subset (\psi \circ i)^{-1}[V'] = i^{-1}[\psi^{-1}[V']]$ implies $\psi^{-1}[V] \not\subset \psi^{-1}[V']$; hence, $\langle Z, \psi \rangle$ is C-distinguishable.

Corollary 1.17

Let $\langle Y, \varphi \rangle$ be an extension of X and $\langle Z, \psi \rangle$ be an extension of Y. Then if $\langle Z, \psi \circ \varphi \rangle$ is a distinguishable, bidistinguishable, or C-distinguishable extension of X, it follows that $\langle Z, \psi \rangle$ is, respectively, a distinguishable, bidistinguishable, or C-distinguishable extension of Y.

Proof: This is a special case of Proposition 1.16 in which the function f is i_Z, the $\langle Y, \varphi \rangle$ is $\langle Z, \psi \circ \varphi \rangle$, and $\langle Z, \psi \rangle$ is $\langle Z, \psi \rangle$.

Corollary 1.18

If $i_1 : X \longrightarrow Y$, $i_2 : Y \longrightarrow W$, and $i_3 : W \longrightarrow Z$ are all dense embeddings, and if $\langle Z, i_3 \circ i_2 \circ i_1 \rangle$ is a distinguishable, bidistinguishable, or C-distinguishable extension of X, then $\langle W, i_2 \rangle$ is, respectively, a distinguishable, bidistinguishable, or C-distinguishable extension of Y.

Proof: From Proposition 1.16, $\langle Z, i_3 \circ i_2 \rangle$ has the associated property and by Proposition 1.12 so does $\langle W, i_2 \rangle$.

Theorem 1.19

If $<Y,\varphi>$ and $<Z,\psi>$ are bidistinguishable extensions of X and if $f:Y \longrightarrow Z$ and $g:Z \longrightarrow Y$ are continuous functions such that $f\circ\varphi = \psi$ and $g\circ\psi = \varphi$, then f and g are homeomorphisms.

Proof: For any $y \in Y$, if $g\circ f(y) \neq y$, then there is some open set U in Y containing $g\circ f(y)$ such that $\varphi^{-1}[U] \nsubseteq \varphi^{-1}[V]$ for any open subset V of Y containing y. But $\varphi^{-1}[U] = (g\circ\psi)^{-1}[U] = \psi^{-1}[g^{-1}[U]] = (f\circ\varphi)^{-1}[g^{-1}[U]] = \varphi^{-1}[f^{-1}[g^{-1}[U]]]$ and $f^{-1}[g^{-1}[U]]$ is an open subset of Y containing y. Hence, $g\circ f(y) = y$ for each $y \in Y$; thus, $g\circ f = 1_Y$. Similarly, $f\circ g = 1_Z$, so that f and g are homeomorphisms.

Corollary 1.20

The class of bidistinguishable extensions of a space X is partially ordered by the relation: $<Y,\varphi> \leq <Z,\psi>$ if there is some continuous function $f:Y \longrightarrow Z$ such that $f\circ\varphi = \psi$. (In Chapter 3 this class will be shown to be a set.)

Proposition 1.21

The conclusion in Theorem 1.19 does not hold for distinguishable or C-distinguishable extensions.

Proof: Let Y be the set $(0,1) \cup \{a,b\}$, where a and b are not elements of $(0,1)$, with topology generated by the base consisting of the open subsets of $(0,1)$, the sets of the form $A \cup \{a\}$ where A is open in $(0,1)$ and contains all but a finite subset of $(0,\frac{1}{2})$, and the sets of the form $B \cup \{a,b\}$ where B is the complement of a finite subset of $(0,1)$. Then if i denotes the inclusion function from $(0,1)$ to Y, $<Y,i>$ is a C-distinguishable extension of $(0,1)$; and there are two distinct continuous functions from Y into Y which

fix elements of $(0,1)$. One of the functions is neither one-to-one nor onto; so it is not a homeomorphism. To show that $\langle Y,i\rangle$ is distinguishable, select two distinct points, x and y, of Y. If one of them (say, x) is an element of $(0,1)$, then $Y \sim \{x\}$ is an open subset of Y containing y. $i^{-1}[Y \sim \{x\}] = (0,1) \sim \{x\}$ and, for any open set U of Y containing x, $x \in i^{-1}[U]$. If, on the other hand, neither is an element of $(0,1)$, then one is a and the other is b. $\{a\} \cup (0,\frac{1}{2})$ is an open set in Y containing a and $i^{-1}[\{a\} \cup (0,\frac{1}{2})] = (0,\frac{1}{2})$. Any open set U in Y containing b contains a set of the form $B \cup \{a,b\}$ where B is the complement of a finite subset of $(0,1)$. Then $B \subseteq i^{-1}[U]$ and B contains elements of $(\frac{1}{2},1)$. Therefore, $\langle Y,i\rangle$ is distinguishable. To show that $\langle Y,i\rangle$ is C-distinguishable, we select any point y and any closed subset A of Y such that $y \notin A$. Three cases must be considered.

<u>Case 1: $y \in (0,1)$</u>. $i^{-1}[Y \sim A]$ is an open subset of $(0,1)$ containing y; so it is possible to choose numbers r and s such that $0 < r < y < s < 1$ and $(r,s) \subseteq i^{-1}[Y \sim A]$. Suppose V is an open set in Y and that $z \in V \cap A$. We consider two subcases.

<u>Subcase (i): $z \in (0,1)$</u>. Since $z \in A$, $z \notin Y \sim A$; so $z \notin i^{-1}[Y \sim A]$. But $z \in i^{-1}[V]$; so $i^{-1}[V] \not\subseteq (r,s) = i^{-1}[(r,s)]$.

<u>Subcase (ii): $z = a$ or $z = b$</u>. Since V is an open set containing either a or b, then all but a finite subset of $(0,r]$ must be contained in V. Hence, $i^{-1}[V] \not\subseteq (r,s) = i^{-1}[(r,s)]$.

<u>Case 2: $y = a$</u>. Since $Y \sim A$ is an open subset of Y containing a, there is some open set of the form $D \cup \{a\}$ contained in $Y \sim A$, where D is the complement of a finite subset in $(0,\frac{1}{2})$. Suppose

V is an open subset of Y and $z \in V \cap A$. We consider two subcases.

<u>Subcase (i): $z \in (0,1)$</u>. Since $z \in A$ and $D \subseteq Y \sim A$, $z \notin D$; so $z \in i^{-1}[V]$ and $z \notin D = i^{-1}[D \cup \{a\}]$. Hence, $i^{-1}[V] \not\subseteq i^{-1}[D \cup \{a\}]$.

<u>Subcase (ii): $z = b$</u>. Since $b \in V$ and V is open in Y, V contains all but a finite subset of $[\frac{1}{2},1)$; so $i^{-1}[V]$ is not a subset of $(0,\frac{1}{2})$ as is $D = i^{-1}[D \cup \{a\}]$. Hence, $i^{-1}[V] \not\subseteq i^{-1}[D \cup \{a\}]$.

<u>Case 3: $y = b$</u>. Since $b \in Y \sim A$ and $Y \sim A$ an open set in Y, A must be a finite subset of $(0,1)$. Suppose V is an open subset of Y containing an element z of A. Then $z \in i^{-1}[V]$ and $z \notin i^{-1}[Y \sim A]$.

Hence, $\langle Y, i\rangle$ is C-distinguishable. Let h be the identity function on Y. Clearly h is a continuous function and $h \circ i = i$. Define f from Y into Y by $f(x) = x$ if $x \in (0,1)$ and $f(a) = f(b) = b$. Again it is clear that $f \circ i = i$. To show that f is continuous, we select $y \in Y$ and an open set U of Y containing $f(y)$ and consider two cases.

<u>Case 1: $y \in (0,1)$</u>. Now $f(y) = y \in U$; so $y \in U \cap (0,1)$ is an open set in Y containing y; thus, $f[U \cap (0,1)] = U \cap (0,1) \subseteq U$.

<u>Case 2: $y \in \{a,b\}$</u>. Now $f(y) = b$; and U contains an open subset of the form $B \cup \{a,b\}$ where B is the complement of a finite subset of $(0,1)$. Then $f[B \cup \{a,b\}] = B \cup \{b\} \subseteq B \cup \{a,b\}$.

Hence, f is continuous. Thus f and h are continuous functions from Y to Y which fix elements of $(0,1)$; but f is clearly not a homeomorphism.

CHAPTER 2

WALLMAN COMPACTIFICATIONS

In Propositions 1.8 and 1.9 it was established that the Wallman compactification (on the lattice of all closed sets) of a T_1 space X has all of the distinguishability properties defined, and is even relatively Hausdorff when X is T_3. Since it retains such an intimate relationship with the structure of the original space, it would seem to deserve further study. From Kelley [5, p. 167] any continuous function from a T_1 space into the unit interval (and, hence, any compact Hausdorff space) can be extended to a continuous function on its Wallman compactification. The similarity between this and the Stone-Čech compactification is so striking as to lead to the conjecture that many of the other properties of the Stone-Čech compactification might have T_1 analogues for the Wallman compactification. The most powerful such result would have a statement something like: If f is a continuous function from a T_1 space X to a compact T_1 space Y, then there is a unique continuous function f^* from $\mathcal{W}(\mathcal{L}_X)$ to Y such that $f = f^* \circ \varphi_{\mathcal{L}_X}$. The following example shows that in some cases there is no such extension, let alone a unique one.

Example 2.1

Let Q denote the rational numbers, and let Q' denote $Q \cup \{\alpha\}$ where α is not a rational number. Let a subset of Q' be open provided that it is an open subset of Q or the complement of a finite subset of Q'. Clearly Q' is compact T_1 and the inclusion function i from Q into Q' is continuous; but i cannot be extended to a continuous function from $\mathcal{W}(\mathcal{L}_Q)$ to Q'.

Proof: If i has a continuous extension i^*, then $i^{*-1}(\alpha)$ is a closed subset of $\mathcal{W}(\mathcal{L}_X)$. For any $u \in \mathcal{W}(\mathcal{L}_Q) \sim \varphi_{\mathcal{L}_Q}[Q]$, $i^*(u) = \alpha$, since if $i^*(u) = x \in Q$, then there exist disjoint open sets U and V in $\mathcal{W}(\mathcal{L}_Q)$ containing $\varphi_{\mathcal{L}_Q}(x)$ and u, respectively (1.9). Then $\varphi_{\mathcal{L}_Q}^{-1}[U]$ and $\varphi_{\mathcal{L}_Q}^{-1}[V]$ are disjoint open sets in Q such that $x \in \varphi_{\mathcal{L}_Q}^{-1}[U]$. Then $i[\varphi_{\mathcal{L}_Q}^{-1}[U]]$ is an open subset of Q' containing x and $\varphi_{\mathcal{L}_Q}^{-1} \circ i^{*-1} \circ i[\varphi_{\mathcal{L}_Q}^{-1}[U]] = \varphi_{\mathcal{L}_Q}^{-1}[U]$, since $i = i^* \circ \varphi_{\mathcal{L}_Q}$. But $u \in i^{*-1} \circ i[\varphi_{\mathcal{L}_Q}^{-1}[U]]$; so $V \cap i^{*-1} \circ i[\varphi_{\mathcal{L}_Q}^{-1}[U]]$ is a nonempty open subset of $\mathcal{W}(\mathcal{L}_Q)$ and, hence, $\emptyset \neq \varphi_{\mathcal{L}_Q}^{-1}[V \cap i^{*-1} \circ i[\varphi_{\mathcal{L}_Q}^{-1}[U]]] \subseteq \varphi_{\mathcal{L}_Q}^{-1}[V]$, which contradicts the fact that $\varphi_{\mathcal{L}_Q}^{-1}[V]$ is disjoint from $\varphi_{\mathcal{L}_Q}[U]$. Therefore, $\mathcal{W}(\mathcal{L}_Q) \sim \varphi_{\mathcal{L}_Q}[Q]$ is closed in $\mathcal{W}(\mathcal{L}_Q)$, which means it is the intersection of some family $\{C(M_\alpha): \alpha \in \mathcal{Q}\}$. However, for any M_α which is a proper subset of Q, $Q \sim M_\alpha$ contains an open interval $(r,s) \cap Q$. Then there exist a and b such that $r < a < b < s$. Since $[a,b] \cap Q$ is not compact, it contains a family of closed sets with the finite intersection property which has empty intersection. By Corollary 0.14, this family is contained in an element u of $\mathcal{W}(\mathcal{L}_Q)$. Then $u \notin C(M_\alpha)$ and $u \notin \varphi_{\mathcal{L}_Q}[X]$. Hence, i cannot be extended.

Happily, however, we can achieve something like the desired result if we restrict our attention to certain well-behaved functions.

Definition 2.2

A continuous function f from a space X to a space Y will be called minutely closed provided that for each ultrafilter u in $\mathcal{L}_X$ there is some $A \in u$ such that for any $B \in u$, $f[B \cap A] \in \mathcal{L}_Y$.

The following is immediate from the definition.

Proposition 2.3

Every continuous closed function is minutely closed. Minutely closed functions have the virtue that they induce functions on Wallman spaces in a very simple way.

Proposition 2.4

If $f: X \longrightarrow Y$ is minutely closed and if u is an ultrafilter in $\mathcal{L}_X$, then $F_f(u)$ is an ultrafilter in $\mathcal{L}_Y$.

Proof: Recall from Definition 0.21 that $F_f(u) = \{A \in \mathcal{L}_Y : f[B] \subseteq A \text{ some } B \in u\}$. From Proposition 0.22, $F_f(u)$ is a filter in $\mathcal{L}_Y$. If $F_f(u)$ is not an ultrafilter in $\mathcal{L}_Y$, then by Proposition 0.12 it is a proper subset of some ultrafilter v in $\mathcal{L}_Y$. Hence, there is some $A \in v \sim F_f(u)$. Since f is continuous, $f^{-1}[A] \in \mathcal{L}_X$. If $f^{-1}[A] \in u$, then $f[f^{-1}[A]] \subseteq A$; so $A \in F_f(u)$. Therefore, $f^{-1}[A] \notin u$; so that from Proposition 0.16, there is some $B \in u$ such that $B \cap f^{-1}[A] = \emptyset$. Since f is minutely closed, there is some $D \in u$ such that $f[D \cap B] \in \mathcal{L}_Y$. But $f[D \cap B] \cap A \subseteq f[B] \cap A = \emptyset$; so A cannot be an element of any filter containing $F_f(u)$. Hence, $F_f(u)$ is an ultrafilter in $\mathcal{L}_Y$.

Remark 2.5

Thus, if $f:X \longrightarrow Y$ is minutely closed, F_f is a function from $\mathcal{W}(\mathcal{L}_X)$ to $\mathcal{W}(\mathcal{L}_Y)$.

Proposition 2.6

If $f:X \longrightarrow Y$ and $g:Y \longrightarrow Z$ are minutely closed, then their composition $g \circ f$ is minutely closed.

Proof: It is well known that the composition of continuous functions is continuous. Let u be an ultrafilter in $\mathcal{L}_X$. By Proposition 2.4, $F_f(u)$ is an ultrafilter in $\mathcal{L}_Y$. Since g is minutely closed, there is some $B \in F_f(u)$ such that for every element A of $F_f(u)$ which is a subset of B, $g[A] \in \mathcal{L}_Z$. Since $B \in F_f(u)$, there is some $D \in u$ such that $f[D] \subseteq B$. Since f is minutely closed, there is some element E of u such that for any $G \in u$, $f[G \cap E] \in \mathcal{L}_Y$. Then for any $G \in u$, $f[G \cap E \cap D] \in \mathcal{L}_Y$ and $f[G \cap E \cap D] \subseteq B$; hence, $g \circ f[G \cap E \cap D] \in \mathcal{L}_Z$; so $g \circ f$ is minutely closed.

Corollary 2.7

The class of topological spaces together with minutely closed functions is a category $\mathcal{M}_C$.

Proposition 2.8

If f is a continuous function from a compact T_1 space X into a T_1 space Y, then f is minutely closed.

Proof: If X is compact and T_1 and if u is an ultrafilter in $\mathcal{L}_X$, then there is some point $x \in X$ such that $\{x\} \in u$. Since Y is T_1, $f[\{z\}] \in \mathcal{L}_Y$ for any $z \in X$. Hence, if $A \in u$, $f[A \cap \{x\}] = f[\{x\}] \in \mathcal{L}_Y$.

Corollary 2.9

If X is a T_1 space and A is a compact subset of X which is not closed, then the inclusion function i of A into X is minutely closed but not closed.

One of the principal virtues of the Stone-Čech compactification is that it induces a functor from the category of completely regular Hausdorff ($T_{3\frac{1}{2}}$) spaces to the category of compact Hausdorff spaces which is a left adjoint to the inclusion functor. Although the Wallman compactification does, as will be shown shortly, induce a functor from $\mathcal{M}_C$ to the category of compact T_1 spaces, there is no hope for making it the left adjoint to the inclusion functor, as the following shows.

Proposition 2.10

If X is a T_1 space which is not compact, then $\varphi_{\mathcal{L}_X}$ is not minutely closed.

Proof: If X is not compact, there is some collection of closed subsets of X which has the finite intersection property and whose intersection is empty. Then by Corollary 0.14, this family is contained in some ultrafilter u in $\mathcal{W}(\mathcal{L}_X)$. Then $u \notin \varphi_{\mathcal{L}_X}[X]$, but for each $A \in u$, $u \in C(A) = \overline{\varphi_{\mathcal{L}_X}[A]}$ (by Definition 0.24 and Proposition 0.35), so $\varphi_{\mathcal{L}_X}[A]$ is not closed in $\mathcal{W}(\mathcal{L}_X)$ for any $A \in u$.

Although the functions $\varphi_{\mathcal{L}_X}$ are not minutely closed, they do share with the minutely closed functions the property of associating ultrafilters to ultrafilters in a unique manner. This association is made precise in the following definition.

Definition 2.11

A continuous function $f:X \longrightarrow Y$ will be called minutely C-preservative provided that for each ultrafilter u in $\mathcal{L}_X$ there is an ultrafilter v_u in $\mathcal{L}_Y$ such that for any open set U in Y which contains an element of v_u there is some element $A_u \in u$ such that $\overline{f[A_u]} \subseteq U$, and if $B \in u$ and $B \subseteq A_u$, then $\overline{f[B]} \in v_u$.

As the next proposition shows, this definition could have been stated much more simply, and possibly in the process might have appeared less artificial. However, as will become evident in the remainder of this chapter, the above form is much better suited to applications.

Proposition 2.12

A continuous function $f:X \longrightarrow Y$ is minutely C-preservative if and only if for each ultrafilter u in $\mathcal{L}_X$ there exists a unique ultrafilter v_u in $\mathcal{L}_Y$ such that $\{\overline{f[A]}:A \in u\} \subseteq v_u$.

Proof: If f is minutely C-preservative, u is an ultrafilter in $\mathcal{L}_X$, and v_u is an ultrafilter in $\mathcal{L}_Y$ (as in 2.11), then for any $A \in u$, $\overline{f[A]} \in v_u$. This is so since Y is an open set containing an element of v_u, which implies that there is some element $B \in u$ such that $\overline{f[B]} \subseteq Y$ and $D \in u$ implies $\overline{f[D \cap B]} \in v_u$; hence, $\overline{f[A \cap B]} \in v_u$ and $\overline{f[A \cap B]} \subseteq \overline{f[A]}$; so $\overline{f[A]} \in v_u$. v_u is unique since if w is any other ultrafilter in $\mathcal{L}_Y$, from Proposition 0.15 there exist disjoint A and B in $\mathcal{L}_Y$ such that $A \in v_u$ and $B \in w$. Hence, $Y \sim B$ is an open subset of Y containing A, a member of v_u. Since f is minutely C-preservative, there is some element D of u such that $\overline{f[D]} \subseteq Y \sim B$ and $\overline{f[D]} \in v_u$; so $\{\overline{f[A]}:A \in u\}$ is not contained in w. Conversely, if for each ultrafilter u in $\mathcal{L}_X$, v_u is the

unique ultrafilter in $\mathcal{L}_Y$ which contains $\{\overline{f[A]}:A \in u\}$, then given any open set U containing an element of v_u, $\{\overline{f[A]} \cap (Y \sim U):A \in u\}$ does not have the finite intersection property. If it did, then by Corollary 0.14 it would be contained in an ultrafilter in $\mathcal{L}_Y$ which would have to be distinct from v_u and contain $\{\overline{f[A]}:A \in u\}$. But then there is some finite subset $\{A_i:i = 1,2,,n\}$ of u such that $(Y \sim U) \cap (\bigcap_{i=1}^{n} \overline{f[A_i]}) = \emptyset$; so $\overline{f[\bigcap_{i=1}^{n} A_i]} \subseteq \bigcap_{i=1}^{n} \overline{f[A_i]} \subseteq U$; and since for any $B \in u$, $B \cap (\bigcap_{i=1}^{n} A_i) \in u$, $\overline{f[B \cap (\bigcap_{i=1}^{n} A_i)]} \in v_u$.

The uniqueness of v_u allows us to define the following.

Definition 2.13

If $f:X \longrightarrow Y$ is minutely C-preservative, define $\hat{f}: \mathcal{W}(\mathcal{L}_X) \longrightarrow \mathcal{W}(\mathcal{L}_Y)$ by $\hat{f}(u) = v_u$, where v_u is the (unique) ultrafilter of Definition 2.11.

Proposition 2.14

If $f:X \longrightarrow Y$ is minutely closed, then f is minutely C-preservative and $\hat{f}$ is continuous.

Proof: Clearly from Propositions 2.4 and 2.12, $F_f(u) = \hat{f}(u)$ for any $u \in \mathcal{W}(\mathcal{L}_X)$. Let v be an element of $\mathcal{W}(\mathcal{L}_X)$ and let V be an open set of Y containing some element B of $F_f(u)$. Since there is some $A \in u$ such that $f[A] \subseteq B$, and since f is continuous, $f^{-1}[B] \in u$. Now for any $D \in u$, $f[D \cap f^{-1}[B]] \subseteq B$, so $\overline{f[D \cap f^{-1}[B]]} \subseteq \overline{B} = B \subseteq V$; so since $\overline{f[D \cap f^{-1}[B]]}$ is clearly an element of $F_f(u)$, f is minutely C-preservative. Let u be an element of $\mathcal{W}(\mathcal{L}_X)$ and let $F_f(u)$ be an element of an open set U in $\mathcal{W}(\mathcal{L}_Y)$. Since the collection $\{C(M):M \in \mathcal{L}_Y\}$ is a base for the closed sets in $\mathcal{W}(\mathcal{L}_Y)$ and $\mathcal{W}(\mathcal{L}_Y) \sim U$ is a closed set in $\mathcal{W}(\mathcal{L}_Y)$ which does not contain $F_f(u)$, there is some $M \in \mathcal{L}_Y$ such that

$F_f(u) \in \mathcal{W}(\mathcal{L}_Y) \sim C(M) \subseteq U$. Since f is continuous, $f^{-1}[M] \in \mathcal{L}_X$; and $f^{-1}[M] \notin u$, since $f^{-1}[M] \in u$ implies $M \in F_f(u)$. By Proposition 0.16, for each ultrafilter w in $\mathcal{W}(\mathcal{L}_X) \sim C(f^{-1}[M])$, there is some $A_w \in w$ such that $A_w \cap f^{-1}[M] = \emptyset$. Since f is minutely closed, there is some $B_w \in w$ such that $D \in w$ implies $f[D \cap B_w]$ is closed; so $\overline{f[A_w \cap B_w]} = f[A_w \cap B_w] \subseteq f[A_w] \subseteq Y \sim M$. Hence, $F_f(w) \in \mathcal{W}(\mathcal{L}_Y) \sim C(M)$; so $u \in \mathcal{W}(\mathcal{L}_X) \sim C(f^{-1}[M])$ and $\hat{f}[\mathcal{W}(\mathcal{L}_X) \sim C(f^{-1}[M])] \subseteq \mathcal{W}(\mathcal{L}_Y) \sim C(M) \subseteq U$. Consequently, $\hat{f}$ is continuous.

Proposition 2.15

If $f:X \longrightarrow Y$, $g:Y \longrightarrow Z$, and $gof:X \longrightarrow Z$ are all minutely C-preservative, then $\widehat{gof} = \hat{g}o\hat{f}$.

Proof: Recall from Definition 2.13 that $\widehat{gof}(u)$ is the (unique) ultrafilter v in $\mathcal{L}_Z$ which contains $\{\overline{gof[A]}:A \in u\}$. $\hat{g}o\hat{f}(u)$ is the (unique) ultrafilter w in $\mathcal{L}_Z$ which contains $\{\overline{g[B]}:B \in \hat{f}(u)\}$. But for every A in u, $\overline{f[A]} \in \hat{f}(u)$ and $\overline{g[f[A]]} = \overline{g[\overline{f[A]}]}$; so $\{\overline{gof[A]}:A \in u\} \subseteq \{\overline{g[B]}:B \in \hat{f}(u)\}$ and, from the uniqueness of w and v, we have $w = v$.

This yields the following obvious corollary.

Corollary 2.16

" $\wedge$ " defines a functor from $\mathcal{M}_C$ to the category of compact T_1 spaces with continuous functions.

For T_1 spaces, " $\wedge$ " has an interesting uniqueness property.

Proposition 2.17

If $f:X \longrightarrow Y$ is minutely C-preservative, if X and Y are T_1 spaces, and if $\varphi_{\mathcal{L}_X}$ and $\varphi_{\mathcal{L}_Y}$ denote the injections of X and Y into $\mathcal{W}(\mathcal{L}_X)$ and $\mathcal{W}(\mathcal{L}_Y)$, respectively, then $\hat{f}o\,\varphi_{\mathcal{L}_X} = \varphi_{\mathcal{L}_Y}of$;

that is, the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[d, "\varphi_{\mathcal{L}_X}"'] & & Y \arrow[d, "\varphi_{\mathcal{L}_Y}"] \\ \mathcal{W}(\mathcal{L}_X) \arrow[rr, "\hat{f}"'] & & \mathcal{W}(\mathcal{L}_Y) \end{tikzcd}$$

Furthermore, if $g: \mathcal{W}(\mathcal{L}_X) \longrightarrow \mathcal{W}(\mathcal{L}_Y)$ is any continuous function for which $g \circ \varphi_{\mathcal{L}_X} = \varphi_{\mathcal{L}_Y} \circ f$, then $g = \hat{f}$.

<u>Proof</u>: For any $x \in X$, $\{x\} \in \{A \in \mathcal{L}_X : x \in A\} = \varphi_{\mathcal{L}_X}(x)$; so from Definition 2.11, $\{f(x)\} \in \hat{f} \circ \varphi_{\mathcal{L}_X}(x)$; but the only element of $\mathcal{W}(\mathcal{L}_Y)$ which contains $\{f(x)\}$ is $\varphi_{\mathcal{L}_Y}(f(x)) = \varphi_{\mathcal{L}_Y} \circ f(x)$. If $g: \mathcal{W}(\mathcal{L}_X) \longrightarrow \mathcal{W}(\mathcal{L}_Y)$ is a continuous function which makes the diagram commute, and if for some $u \in \mathcal{W}(\mathcal{L}_X)$, $g(u) \neq \hat{f}(u)$, then there exist disjoint $M_1, M_2 \in \mathcal{L}_Y$ such that $M_1 \in g(u)$ and $M_2 \in \hat{f}(u)$. Then since $Y \sim M_1$ is an open subset of Y containing M_2 (an element of $\hat{f}(u)$), there is some $A \in u$ such that $\overline{f[A]} \in \hat{f}(u)$ and $\overline{f[A]} \subseteq Y \sim M_1$. Since M_1 and $\overline{f[A]}$ are disjoint elements of $\mathcal{L}_Y$, $C(M_1)$ and $C(\overline{f[A]})$ are disjoint closed subsets of $\mathcal{W}(\mathcal{L}_Y)$ and $g(u) \in C(M_1)$; so $g(u) \notin C(\overline{f[A]})$. Then $u \notin g^{-1}[C(\overline{f[A]})]$. Since g is continuous, $g^{-1}[C(\overline{f[A]})]$ is closed in $\mathcal{W}(\mathcal{L}_X)$; since $C(\overline{f[A]})$ is closed in $\mathcal{W}(\mathcal{L}_Y)$, $A \subseteq f^{-1}[f[A]] \subseteq f^{-1}[\overline{f[A]}]$. From Proposition 0.35, $\overline{f[A]} = \varphi^{-1}_{\mathcal{L}_Y}[C(\overline{f[A]})]$; so $A \subseteq f^{-1}[\varphi^{-1}_{\mathcal{L}_Y}[C(\overline{f[A]})]] = \varphi^{-1}_{\mathcal{L}_X}[g^{-1}[C(\overline{f[A]})]]$; so $\varphi_{\mathcal{L}_X}[A] \subseteq g^{-1}[C(\overline{f[A]})]$, which is closed since g is continuous. Therefore, from Proposition 0.35, $C(A) \subseteq g^{-1}[C(\overline{f[A]})]$; hence, $u \in g^{-1}[C(\overline{f[A]})]$--a contradiction. Thus, $g = \hat{f}$.

Corollary 2.18

$\hat{f} = \hat{\hat{f}}$.

It has been established that " $\wedge$ " determines a functor on $\mathcal{M}_C$, a category which does not contain the Wallman injections. " $\wedge$ " has been defined for all minutely C-preservative functions. It will be shown that each Wallman injection, $\varphi_{\mathcal{L}_X}$, is minutely C-preservative and that $\widehat{\varphi_{\mathcal{L}_X}}$ is continuous. Although Proposition 2.17 shows that for any minutely C-preservative function f, $\hat{f}$ is the only function which commutes with the Wallman injections which can be continuous, it has been impossible to date to determine whether $\hat{f}$ is necessarily continuous. It has also been impossible to determine whether the composition of minutely C-preservative functions is minutely C-preservative. Proposition 2.12 would seem to indicate that minutely C-preservative is the weakest possible condition for the existence of a unique function which commutes with the Wallman injections. It is also possible to define intuitively minimal conditions to require that $\hat{f}$ be continuous (namely, for each ultrafilter u in $\mathcal{L}_X$ there is a unique ultrafilter v_u in $\mathcal{L}_Y$ such that for any open set U in Y which contains an element of v_u, there is an open set V_u in X containing an element of u which has the property that for any ultrafilter w in $\mathcal{L}_X$ which does not contain $X \sim V_u$, there is some element $A \in w$ such that $\overline{f[A]} \subseteq U$). Similarly, one can define intuitively minimal conditions for the composition of any two functions to be minutely C-preservative (namely, for any filter $\mathcal{F}$ in $\mathcal{L}_X$ such that $\bigcap_{A \in \mathcal{F}} C(A)$ is a singleton, $\bigcap_{A \in \mathcal{F}} C(\overline{f[A]})$ is a singleton). The following is an attempt to incorporate both of the above conditions into one definition to

yield a class of functions which is closed under composition and which has continuous functions as images under " $\wedge$ ".

Definition 2.19

A minutely C-preservative function $f:X \longrightarrow Y$ will be called properly closed provided that given any $u \in \mathcal{W}(\mathcal{L}_X)$ and any open V in Y containing an element of $\hat{f}(u)$ there exists an open set U_V in X containing an element of u, such that, for any $A \in \mathcal{L}_X$, $A \subseteq U_V \Longrightarrow \overline{f[A]} \subseteq V$.

Proposition 2.20

If $f:X \longrightarrow Y$ and $g:Y \longrightarrow Z$ are properly closed, then $g \circ f$ is properly closed.

Proof: Given any $u \in \mathcal{W}(\mathcal{L}_X)$ and any open V in Z containing an element of $\hat{g}\circ\hat{f}(u)$, there is some open U_V in Y containing an element of $\hat{f}(u)$ such that for any $A \in \mathcal{L}_Y$, $A \subseteq U_V \Longrightarrow \overline{g[A]} \subseteq V$. Since f is properly closed, there is some open W_{U_V} in X containing an element of u such that for any $B \in \mathcal{L}_X$, $B \subseteq W_{U_V} \Longrightarrow \overline{f[B]} \subseteq U_V \Longrightarrow \overline{g\circ f[B]} = \overline{g[f[B]]} \subseteq V$.

Corollary 2.21

The class $\mathcal{P}_C$ of topological spaces together with properly closed functions is a category.

Proposition 2.22

If $f:X \longrightarrow Y$ is properly closed, then $\hat{f}$ is properly closed.

Proof: Let U be an open subset of $\mathcal{W}(\mathcal{L}_Y)$ and let $\hat{f}(u)$ be an element of U. Since the collection $\{C(M):M \in \mathcal{L}_Y\}$ is a base for the closed sets in $\mathcal{W}(\mathcal{L}_Y)$, there is some element $M_0 \in \mathcal{L}_Y$ such that $\hat{f}(u) \in \mathcal{W}(\mathcal{L}_Y) \sim C(M_0) \subseteq U$. By Proposition 0.16, there is some element $M_1 \in \hat{f}(u)$ such that $M_1 \cap M_0 = \emptyset$. Since f is properly

closed, there is some open set V in X which contains an element of u such that for any closed set $A \subseteq V$, $\overline{f[A]} \subseteq Y \sim M_0$. Hence, for any $v \in \mathcal{W}(\mathcal{L}_X) \sim C(X \sim V)$ there is some element $A_v \in v$ such that $A_v \cap (X \sim V) = \emptyset$ (0.16) and $\overline{f[A_v]} \in \hat{f}(v)$ (2.12); so $\hat{f}(v) \in C(\overline{f[A_v]})$ (0.24) and $C(\overline{f[A_v]}) \cap C(M_0) = \emptyset$ (0.26 and 0.27). Consequently, $\hat{f}(v) \in \mathcal{W}(\mathcal{L}_Y) \sim C(M_0) \subseteq U$. Thus, $\hat{f}$ is continuous. Since $\hat{f}$ is a continuous function between compact T_1 spaces, $\hat{f}$ is minutely C-preservative (2.8 and 2.14). Finally we must show that given $u \in \mathcal{W}(\mathcal{L}_{\mathcal{W}(\mathcal{L}_X)})$ (u must be of the form $\{A \in \mathcal{L}_{\mathcal{W}(\mathcal{L}_X)} : u' \in A\}$ for some $u' \in \mathcal{W}(\mathcal{L}_X)$) and any open set V in $\mathcal{W}(\mathcal{L}_Y)$ containing an element of $\hat{\hat{f}}(u)$ (which, of course, is $\{A \in \mathcal{L}_{\mathcal{W}(\mathcal{L}_Y)} : \hat{f}(u') \in A\}$ so V is any open set containing $\hat{f}(u')$), there is some open set U_V in $\mathcal{W}(\mathcal{L}_X)$ containing an element of u (and, hence, u') such that for any closed subset B of U_V, $\overline{\hat{f}[B]} \subseteq V$. Since V is an open subset of $\mathcal{W}(\mathcal{L}_Y)$, $\mathcal{W}(\mathcal{L}_Y) \sim V$ is closed; so $\mathcal{W}(\mathcal{L}_Y) \sim V = \bigcap_{\alpha \in \mathcal{A}} C(M_\alpha)$ for some collection $\{M_\alpha : \alpha \in \mathcal{A}\} \subseteq \mathcal{L}_Y$. Hence, there is some $M_0 \in \mathcal{L}_Y$ such that $\hat{f}(u') \in \mathcal{W}(\mathcal{L}_Y) \sim C(M_0) \subseteq V$. Then since $M_0 \notin \hat{f}(u')$, there is some $M_1 \in \hat{f}(u')$ such that $M_1 \subseteq Y \sim M_0$; so $Y \sim M_0$ is an open subset of Y containing an element of $\hat{f}(u')$. Since f is properly closed, there is some open set W in X containing an element A of u' such that for any closed subset B contained in W, $\overline{f[B]} \subseteq Y \sim M_0$. Then $\mathcal{W}(\mathcal{L}_X) \sim C(X \sim W)$ is an open subset of $\mathcal{W}(\mathcal{L}_X)$. $u' \in C(A)$ and $A \cap (X \sim W) = \emptyset$ implies $u' \in C(A) \subseteq \mathcal{W}(\mathcal{L}_X) \sim C(X \sim W)$; so $C(A)$ is an element of u contained in $\mathcal{W}(\mathcal{L}_X) \sim C(X \sim W)$. If D is any closed subset of $\mathcal{W}(\mathcal{L}_X)$ contained in $\mathcal{W}(\mathcal{L}_X) \sim C(X \sim W)$, then $D = \bigcap_{\beta \in \mathcal{B}} C(M_\beta)$ for some collection $\{M_\beta : \beta \in \mathcal{B}\} \subseteq \mathcal{L}_X$; so since

$\{C(X \sim W)\} \cup \{C(M_\beta): \beta \in \mathcal{B}\}$ is a collection of closed subsets of a compact space with empty intersection, there is some finite subset θ of $\mathcal{B}$ such that $\bigcap_{\beta \in \theta} C(M_\beta) \subseteq C(X \sim W)$. Then $\overline{\hat{f}[D]} \subseteq \overline{\hat{f}[\bigcap_{\beta\in\theta} C(M_\beta)]} = \overline{\hat{f}[C(\bigcap_{\beta\in\theta} M_\beta)]}$. If $v \in C(\bigcap_{\beta\in\theta} M_\beta)$, $\bigcap_{\beta\in\theta} M_\beta \in v$; so by Proposition 2.12, $\overline{f[\bigcap_{\beta\in\theta} M_\beta]} \in \hat{f}(v)$. Hence, $\overline{\hat{f}[C(\bigcap_{\beta\in\theta} M_\beta)]} \subseteq C(\overline{f[\bigcap_{\beta\in\theta} M_\beta]})$. But since $\bigcap_{\beta\in\theta} M_\beta$ is closed in X and contained in W, $\overline{f[\bigcap_{\beta\in\theta} M_\beta]} \subseteq Y \sim M_0$; so $C(\overline{f[\bigcap_{\beta\in\theta} M_\beta]}) \subseteq \mathcal{W}(\mathcal{L}_Y) \sim C(M_0) \subseteq V$. Thus, $\overline{\hat{f}[B]} \subseteq V$; so $\hat{f}$ is properly closed.

Proposition 2.23

If X is a T_1 space, then $\varphi_{\mathcal{L}_X}$ is properly closed and $\hat{\varphi}_{\mathcal{L}_X}$ is a homeomorphism.

Proof: By Proposition 0.36, $\varphi_{\mathcal{L}_X}$ is continuous. If u is any ultrafilter in $\mathcal{L}_X$, let v_u denote the ultrafilter in $\mathcal{L}_{\mathcal{W}(\mathcal{L}_X)}$ generated by $\{u\}$. Clearly, if an open set U contains an element of v_u, it will contain $\{u\}$. $\mathcal{W}(\mathcal{L}_X) \sim U$ is closed in $\mathcal{W}(\mathcal{L}_X)$; so it is the intersection of sets of the form C(M) with $M \in \mathcal{L}_X$. Hence, there is some $M \in \mathcal{L}_X$ such that $u \in \mathcal{W}(\mathcal{L}_X) \sim C(M) \subseteq U$. Then $X \sim M$ is an open set in X and there is some element of u contained in $X \sim M$ (since otherwise $M \in u$ (0.16) and, consequently, $u \in C(M)$). If A is a closed set in X contained in $X \sim M$, then $\overline{\varphi_{\mathcal{L}_X}[A]} = C(A)$ (0.35); but since $A \cap M = \emptyset$, $C(A) \cap C(M) = \emptyset$ (0.26 and 0.27). Hence, $\overline{\varphi_{\mathcal{L}_X}[A]} = C(A) \subseteq \mathcal{W}(\mathcal{L}_X) \sim C(M) \subseteq U$. Thus, $\varphi_{\mathcal{L}_X}$ is minutely C-preservative. $\varphi_{\mathcal{L}_X}$ is also properly closed since we have taken an arbitrary open set U containing an element of $v_u (= \hat{\varphi}_{\mathcal{L}_X}(u)$ (2.11)), found an open set V in X containing

an element of u, and shown that for any closed subset A of X contained in V the closure of $\varphi_{\mathcal{L}_X}[A]$ is contained in U.

Obviously $\varphi_{\mathcal{L}_{\mathcal{W}(\mathcal{L}_X)}}$ is a continuous function for which the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "\varphi_{\mathcal{L}_X}"] \arrow[d, "\varphi_{\mathcal{L}_X}"'] & \mathcal{W}(\mathcal{L}_X) \arrow[d, "\varphi_{\mathcal{L}_{\mathcal{W}(\mathcal{L}_X)}}"] \\
\mathcal{W}(\mathcal{L}_X) \arrow[r, dashed] & \mathcal{W}(\mathcal{L}_{\mathcal{W}(\mathcal{L}_X)})
\end{tikzcd}$$

From Proposition 2.17, $\hat{\varphi}_{\mathcal{L}_X}$ is the only function which can have this property. Hence, $\hat{\varphi}_{\mathcal{L}_X} = \varphi_{\mathcal{L}_{\mathcal{W}(\mathcal{L}_X)}}$, which is a homeomorphism by Corollary 0.37.

Proposition 2.24

If $f: X \longrightarrow Y$ is continuous and Y is normal, then f is properly closed.

Proof: Let u be an ultrafilter in $\mathcal{L}_X$, and let v_u denote an arbitrary ultrafilter in $\mathcal{L}_Y$ containing $F_f(u)$. If M is any element of $\mathcal{L}_Y \sim v_u$, then by Proposition 0.16 there is some element $A \in v_u$ such that $M \cap A = \emptyset$. Since Y is normal, there exist disjoint open sets U and V in Y such that $A \subseteq U$ and $M \subseteq V$. Then
$X = f^{-1}[Y] = f^{-1}[(Y \sim U) \cup (Y \sim V)] = f^{-1}[Y \sim U] \cup f^{-1}[Y \sim V]$;
so $f^{-1}[Y \sim U]$ and $f^{-1}[Y \sim V]$ are elements of $\mathcal{L}_X$. By Proposition 0.17, $f^{-1}[Y \sim U] \in u$ or $f^{-1}[Y \sim V] \in u$. If $f^{-1}[Y \sim U] \in u$, then $f[f^{-1}[Y \sim U]] \subseteq Y \sim U$ implies $Y \sim U \in F_f(u) \subseteq v_u$. But $A \cap [Y \sim U] = \emptyset$ implies that $Y \sim U \notin v_u$. Consequently, $f^{-1}[Y \sim V] \in u$. Then, for any $B \in u$, $f[B \cap f^{-1}[Y \sim V]] \subseteq Y \sim V \subseteq Y \sim M$;

so $\overline{f[B \cap f^{-1}[Y \sim V]]} \subseteq Y \sim V \subseteq Y \sim M$ and $\overline{f[B \cap f^{-1}[Y \sim V]]} \in v_u$. Hence, f is minutely C-preservative. Since M and $Y \sim V$ are disjoint closed subsets of Y, there exist disjoint open sets W_1 and W_2 of Y such that $M \subseteq W_1$ and $Y \sim V \subseteq W_2$. Then $f^{-1}[W_2]$ is an open subset of X which contains $f^{-1}[Y \sim V]$, an element of u. If D is any closed subset of X contained in $f^{-1}[W_2]$, then $f[D] \subseteq W_2 \subseteq Y \sim W_1 \subseteq Y \sim M$; so $\overline{f[D]} \subseteq Y \sim W_1 \subseteq Y \sim M$. Hence, f is properly closed.

Corollary 2.25

The category $\mathcal{T}op_N$ of normal spaces and continuous functions is a full subcategory of $\mathcal{P}_C$.

Theorem 2.26

The Wallman compactification (on the full lattice of all closed sets) induces a functor $\mathcal{W}$ from the category $\mathcal{P}_C^{(1)}$ of T_1 spaces and properly closed functions to the category $\mathcal{CP}_C^{(1)}$ of compact T_1 spaces and properly closed functions such that the inclusion functor of $\mathcal{CP}_C^{(1)}$ into $\mathcal{P}_C^{(1)}$ is a right adjoint for $\mathcal{W}$. Thus the category $\mathcal{CP}_C^{(1)}$ is reflective in $\mathcal{P}_C^{(1)}$. Furthermore, on the category of T_4 spaces, $\mathcal{W}$ is precisely the Stone-Čech functor β.

Proof: That $\mathcal{W}(f) = \hat{f}$ defines a functor from $\mathcal{P}_C^{(1)}$ to $\mathcal{CP}_C^{(1)}$ is immediate from Propositions 2.15, 2.20, and 2.22. That $\mathcal{W}$ is a left adjoint to the inclusion functor follows from Corollary 0.37, Proposition 2.17, and Proposition 2.23. $\mathcal{W}$ and β agree on T_4 spaces because the Stone-Čech injection $e: X \longrightarrow \beta(X)$ is properly closed, by Proposition 2.24. So $\hat{e}$ is a continuous function from $\mathcal{W}(\mathcal{L}_X)$ to $\mathcal{W}(\mathcal{L}_{\beta X})$. $\mathcal{W}(\mathcal{L}_{\beta X})$ is homeomorphic with βX, by Corollary 0.37. $\mathcal{W}(\mathcal{L}_X)$ is Hausdorff by Proposition 0.30; so there

is a unique function η from βX to $\mathcal{W}(\mathcal{L}_X)$ such that $\eta \circ e = \varphi_{\mathcal{L}_X}$. Hence, since both $<\beta X, e>$ and $<\mathcal{W}(\mathcal{L}_X), \varphi_{\mathcal{L}_X}>$ are bidistinguishable extensions of X, $\hat{e}$ and η are homeomorphisms (1.19).

CHAPTER 3

DIRECT LIMITS

In contrast to the collection of ultrafilters in the lattice, $\mathcal{L}_X$, of closed sets of a space X, Strauss [6] has used a corresponding structure of ultrafilters in the lattice $\mathcal{T}_X$ of open sets to study the structure of, and relationships among, T_3 spaces. From the collection of all ultrafilters in $\mathcal{T}_X$, which with appropriate topology is compact, Hausdorff, and extremally disconnected (i.e., the closure of each open set is open), she selects a dense subspace E(X) and defines a function $r:E(X) \longrightarrow X$. This function is continuous, closed, onto, and compact (the inverse image of each point is compact in E(X)). Furthermore, Strauss has shown that $r:E(X) \longrightarrow X$ is a "projective resolution" in the category of T_3 spaces and perfect (i.e., continuous, closed, and compact) maps. More precisely, the restriction of r to no proper closed subset of E(X) is surjective and if $f:Y \longrightarrow X$ is any perfect epimorphism in the category, then there is a perfect map $f':E(X) \longrightarrow Y$ such that $r = f \circ f'$. r is ordinarily many-to-one, so E(X) is not an extension of X by any means, but if the structure is enlarged somewhat, we can achieve an extension with several very useful properties. It will, for instance, be a C-distinguishable extension,

which contains as a subspace a homeomorphic image of every C-distinguishable extension of X. It also serves as a "range" for maps in establishing the preservation of the Hausdorff separation property under direct limits.

Definition 3.1

For a topological space $(X, \mathcal{T}_X)$, define $\mathcal{O}(X)$ to be the collection of all filters in $\mathcal{T}_X$, and for each $U \in \mathcal{T}_X$, define $U^{\#} = \{\mathcal{F} \in \mathcal{O}(X) : U \in \mathcal{F}\}$.

Proposition 3.2

$\{U^{\#} : U \in \mathcal{T}_X\}$ is a base for a topology on $\mathcal{O}(X)$.

Proof: For any $\mathcal{F} \in \mathcal{O}(X)$, $X \in \mathcal{F}$ and $X \in \mathcal{T}_X$; so $\mathcal{F} \in X^{\#}$. If $\mathcal{F} \in U^{\#} \cap V^{\#}$, then $U \in \mathcal{F}$ and $V \in \mathcal{F}$; so $U \cap V \in \mathcal{F}$ which means $\mathcal{F} \in (U \cap V)^{\#}$. But, conversely, if $\mathcal{F} \in (U \cap V)^{\#}$, then $(U \cap V) \in \mathcal{F}$ and $(U \cap V) \subseteq U$ and $(U \cap V) \subseteq V$; so $U \in \mathcal{F}$ and $V \in \mathcal{F}$. Hence, $\mathcal{F} \in U^{\#} \cap V^{\#}$.

Henceforth, $\mathcal{O}(X)$ will be assumed to have this topology.

Definition 3.3

For any extension $\langle Y, \varphi \rangle$ of X and any $y \in Y$, let $g_{\varphi}(y) = \{U \in \mathcal{T}_X : \varphi^{-1}[V] \subseteq U$ for some open V in Y containing $y\}$.

Proposition 3.4

For any extension $\langle Y, \varphi \rangle$ of X, g_{φ} as defined above is a continuous function from Y to $\mathcal{O}(X)$.

Proof: For any point $y \in Y$, $g_{\varphi}(y)$ is a filter in $\mathcal{T}_X$ since:

$g_{\varphi}(y)$ is defined to be a subset of $\mathcal{T}_X$ and, since Y is an open subset of Y containing y and $X \subseteq X = \varphi^{-1}[Y]$, $g_{\varphi}(y)$ is nonempty.

(i) If $\emptyset \in g_\varphi(y)$, then there is some open set U in Y containing y such that $\varphi^{-1}[U] = \emptyset$. But this would mean that $\varphi[X]$ is not dense in Y; hence, $<Y,\varphi>$ is not an extension.

(ii) If A and B are elements of $g_\varphi(y)$, then there are open sets U and V in Y containing y such that $\varphi^{-1}[U] \subseteq A$ and $\varphi^{-1}[V] \subseteq B$. Then $U \cap V$ is an open subset of Y containing y, and $\varphi^{-1}[U \cap V] = \varphi^{-1}[U] \cap \varphi^{-1}[V] \subseteq A \cap B$; so $A \cap B \in g_\varphi(y)$.

(iii) If $A \in g_\varphi(y)$ and if B is an element of $\mathcal{T}_X$ containing A, then there is some open U in Y containing y such that $\varphi^{-1}[U] \subseteq A \subseteq B$. Thus, $B \in g_\varphi(y)$. Hence, $g_\varphi(y) \in \mathcal{O}(X)$ for each $y \in Y$. If U is an open subset of $\mathcal{O}(X)$ containing $g_\varphi(y)$, then, since $\{V^\#: V \in \mathcal{T}_X\}$ is a base for the topology on $\mathcal{O}(X)$, there is some $V \in \mathcal{T}_X$ such that $g_\varphi(y) \in V^\# \subseteq U$. Since $g_\varphi(y) \in V^\#$, $V \in g_\varphi(y)$, which implies there is some open set W in Y containing y such that $\varphi^{-1}[W] \subseteq V$. Thus for any $w \in W$, $\varphi^{-1}[W] \in g_\varphi(w)$; so $g_\varphi(w) \in V^\#$. Hence, $y \in W \subseteq g_\varphi^{-1}[V^\#] \subseteq g_\varphi^{-1}[U]$, so g_φ is continuous.

<u>Proposition 3.5</u>

If $<Y,\varphi>$ is a distinguishable extension of X, then g_φ is one-to-one.

<u>Proof</u>: Given two points y and z of Y, there is some open U containing one (say, y) such that $\varphi^{-1}[U] \not\subseteq \varphi^{-1}[V]$ for any open V containing the other. Then $g_\varphi(y) \in (\varphi^{-1}[U])^\#$. If $g_\varphi(z) \in (\varphi^{-1}[U])^\#$, there is some open W in Y containing z such that $\varphi^{-1}[W] \subseteq \varphi^{-1}[U]$. But then $z \in W \cup U$ and $\varphi^{-1}[W \cup U] = \varphi^{-1}[U]$, which contradicts the choice of U. Therefore, $g_\varphi(z) \notin (\varphi^{-1}[U])^\#$; so that $g_\varphi(z) \neq g_\varphi(y)$.

This, then, means that a distinguishable extension of X is essentially a subset of the set $\mathcal{O}(X)$ with some topology; so the collection of distinguishable extensions of X is a subset of $\mathcal{P}(\mathcal{O}(X)) \times \mathcal{P}(\mathcal{P}(\mathcal{O}(X)))$.

Thus, we have shown:

<u>Corollary 3.6</u>

The class of distinguishable extensions of a space, X, is a set.

<u>Proposition 3.7</u>

If $\langle Y, \varphi \rangle$ is a C-distinguishable extension of X, then g_φ is relatively open.

<u>Proof</u>: If U is an open subset of Y and if $y \in U$, then there is some open V in Y such that $y \in V$ and $\varphi^{-1}[V] \not\subseteq \varphi^{-1}[W]$ for any open subset W of Y for which $W \cap (Y \sim U) \neq \emptyset$. Then $g_\varphi(y) \in (\varphi^{-1}[V])^{\#}$. If for any $z \in (Y \sim U)$, $g_\varphi(z) \in (\varphi^{-1}[V])^{\#}$, then there is some open $W \subseteq Y$ such that $z \in W$ and $\varphi^{-1}[W] \subseteq \varphi^{-1}[V]$; but this would mean $z \in (W \cup V)$ and $\varphi^{-1}[W \cup V] = \varphi^{-1}[V]$, which contradicts the choice of V. Hence, $g_\varphi(y) \in (\varphi^{-1}[V])^{\#} \cap g_\varphi[Y] \subseteq g_\varphi[U]$.

The above proposition shows that the C-distinguishable extensions of X can be thought of as subspaces of $\mathcal{O}(X)$. The converse, that every subspace of $\mathcal{O}(X)$ (containing the image of X) is a C-distinguishable extension of X, follows from the next proposition and Proposition 1.12.

<u>Proposition 3.8</u>

If X is a T_0 space and i is the identity function on X, then $\langle \mathcal{O}(X), g_i \rangle$ is a C-distinguishable extension of X.

Proof: For any two points in X, there is an open set containing one and not the other. Also, for any two open subsets U and V of X, $i^{-1}[U] = i^{-1}[V]$ if and only if $U = V$. Thus, $<X,i>$ is a C-distinguishable extension of X and, by Propositions 3.4, 3.5, and 3.7, g_i is an embedding. If U is any nonempty open subset of $\mathcal{O}(X)$, then there is some nonempty $V^{\#} \subseteq U$. Then V is nonempty since $\mathcal{F} \in V^{\#}$ implies $V \in \mathcal{F}$, and $\emptyset \notin \mathcal{F}$ for every $\mathcal{F} \in \mathcal{O}(X)$. Then there is some $x \in V = i^{-1}[V]$; so $g_i(x) \in V^{\#}$. Hence, $g_i[X]$ is dense in $\mathcal{O}(X)$; so $<\mathcal{O}(X),g_i>$ is an extension of X. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be distinct filters in $\mathcal{J}_X$. Since they are distinct, there is some element $V \in \mathcal{J}_X$ which is an element of one (say, $\mathcal{F}_1$) and not the other. Then $\mathcal{F}_1 \in V^{\#}$ and $g_i^{-1}[V^{\#}] = V$. If there is some open set W in $\mathcal{O}(X)$ containing $\mathcal{F}_2$ such that $g_i^{-1}[W] = V$, then there is some basic open set $U^{\#}$ in $\mathcal{O}(X)$ such that $\mathcal{F}_2 \in U^{\#} \subseteq W$. Then $U = g_i^{-1}[U^{\#}] \subseteq g_i^{-1}[W] = V$. But since $\mathcal{F}_2$ is a filter and $U \in \mathcal{F}_2$, it follows that $V \in \mathcal{F}_2$, which contradicts the choice of V. Hence, no such W exists, so that $<\mathcal{O}(X),g_i>$ is distinguishable. If an element $\mathcal{F}$ of $\mathcal{O}(X)$ is contained in an open set W in $\mathcal{O}(X)$, then there is some basic open set $U^{\#}$ such that $\mathcal{F} \in U^{\#} \subseteq W$. For any $\mathcal{F}' \in \mathcal{O}(X)$, if there is an open set $R \subseteq W$ containing $\mathcal{F}'$ such that $g_i^{-1}[R] = g_i^{-1}[U^{\#}] = U$, then there is some basic open set $S^{\#}$ such that $\mathcal{F}' \in S^{\#} \subseteq R$. Then $S = g_i^{-1}[S^{\#}] \subseteq g_i^{-1}[U^{\#}] = U$. $\mathcal{F}'$ an element of $S^{\#}$ implies by definition that $S \in \mathcal{F}'$; so, since $\mathcal{F}'$ is a filter, $U \in \mathcal{F}'$. Hence, $\mathcal{F}' \in U^{\#}$ if and only if there is some open set V in $\mathcal{O}(X)$ containing $\mathcal{F}'$ such that $g_i^{-1}[V] = U$. Therefore, if T is open in $\mathcal{O}(X)$ and $T \cap (\mathcal{O}(X) \sim W) \neq \emptyset$, then $g_i^{-1}[T] \neq g_i^{-1}[U^{\#}]$; so $<\mathcal{O}(X),g_i>$ is C-distinguishable.

Proposition 3.9

If $<Y, \varphi>$ and $<Z, \psi>$ are extensions of X and if f is an embedding of Y to Z such that $\psi = f \circ \varphi$, then $g_\varphi = g_\psi \circ f$.

Proof: For any $y \in Y$ and $U \in \mathcal{T}_X$, $U \in g_\varphi(y)$ implies that there is some open V in Y containing y such that $\varphi^{-1}[V] \subseteq U$. Since $y \in V$, $f(y) \in f[V] = f[Y] \cap W$ for some open set W in Z. Now $\psi^{-1}[W] = (g \circ \varphi)^{-1}[W] = \varphi^{-1}[f^{-1}[W]] = \varphi^{-1}[V] \subseteq U$; hence $U \in g_\psi \circ f(y)$. Conversely, if $U \in g_\psi \circ f(y)$, then there is some open W in Z containing f(y) such that $\psi^{-1}[W] \subseteq U$. Then $y \in f^{-1}[W]$ open in Y and $\varphi^{-1}[f^{-1}[W]] = (f \circ \varphi)^{-1}[W] = \psi^{-1}[W] \subseteq U$, which implies that $U \in g_\varphi(y)$. Consequently, for each $y \in Y$, $g_\varphi(y) = g_\psi \circ f(y)$.

Dugundji [2] gives an example which shows that even with the nicest of spaces, if, in a direct system, the bonding maps are not one-to-one, the direct limit may be indiscrete. Herrlich [3] has shown that, under the conditions that each bonding map is an embedding and that the image of each space is closed in the direct limit, although the direct limit of T_1 spaces is T_1 and the direct limit of an increasing sequence of T_4 spaces is T_4, the direct limit of completely regular Hausdorff spaces need not be Hausdorff. We now use the properties of $\mathcal{O}(X)$ to establish conditions sufficient for the direct limit of Hausdorff spaces to be Hausdorff.

Theorem 3.10

Let $\{X_\lambda, f_\alpha^\beta, \Lambda\}$ be a direct system of Hausdorff spaces in which each bonding map f_α^β is an embedding. If there is a

space X and a collection $\{h_\lambda : X \longrightarrow X_\lambda \mid \lambda \in \Lambda\}$ of dense embeddings such that $f_\alpha^\beta \circ h_\alpha = h_\beta$ for each $\beta \geq \alpha$ in Λ, then $\varinjlim X_\lambda$ is Hausdorff.

Proof: By Definition 0.8, for each $\lambda \in \Lambda$ $\langle X_\lambda, h_\lambda \rangle$ is an extension of X; so from Propositions 3.4 and 3.9, for each $\lambda \in \Lambda$, g_{h_λ} is a continuous function from X_λ to $\mathcal{O}(X)$ and, if $\alpha \leq \beta$, $g_{h_\alpha} = g_{h_\beta} \circ f_\alpha^\beta$. Hence, by the definition of direct limit, there is a unique continuous function $g : \varinjlim X_\lambda \longrightarrow \mathcal{O}(X)$ such that for each $r \in \Lambda$, $g_{h_r} = g \circ i_r$ (where i_r is the canonical map of X_r to $\varinjlim X_\lambda$). Given any two points a and b of $\varinjlim X_\lambda$, there is some $r \in \Lambda$ and two points x and y of X_r such that $i_r(x) = a$ and $i_r(y) = b$. Since X_r is Hausdorff, there are disjoint open sets U and V in X_r containing x and y, respectively. Then $h_r^{-1}[U] \in g_{h_r}(x)$ and $h_r^{-1}[V] \in g_{h_r}(y)$; so $g_{h_r}(x) \in (h_r^{-1}[U])^\#$ and $g_{h_r}(y) \in (h_r^{-1}[V])^\#$. $(h_r^{-1}[U])^\#$ and $(h_r^{-1}[V])^\#$ are disjoint open sets since they are basic open sets, and $\mathcal{F} \in (h_r^{-1}[U])^\# \cap (h_r^{-1}[V])^\#$ implies that $h_r^{-1}[U] \in \mathcal{F}$ and $h_r^{-1}[V] \in \mathcal{F}$. This means that $h_r^{-1}[U] \cap h_r^{-1}[V] = h_r^{-1}[U \cap V] = h_r^{-1}[\emptyset] = \emptyset \in \mathcal{F}$--a contradiction. Hence, $a \in g^{-1}[(h_r^{-1}[U])^\#]$ and $b \in g^{-1}[(h_r^{-1}[V])^\#]$, which must be disjoint open sets in $\varinjlim X_\lambda$.

The following example due to Herrlich [3] shows that the condition in the theorem that X be densely embedded in each X_α cannot be deleted.

Example 3.11

Let $\{C_n\}$ be a sequence of pairwise disjoint, completely regular, nonnormal spaces, and let (A_n, B_n) be a pair of disjoint closed subsets of C_n which cannot be separated by open sets in C_n. Add to $\bigcup_{i=1}^{\infty} C_n$ two points a and b and topologize the resulting space X

in the following manner:

$$U \text{ open} \iff \begin{cases} U \cap C_n \text{ open for all } n. \\ a \in U \Rightarrow A_n \subseteq U \text{ for all but finitely many } n \\ b \in U \Rightarrow B_n \subseteq U \text{ for all but finitely many } n \end{cases}$$

Then X obviously is not T_2, but it is the direct limit of the $T_{3\frac{1}{2}}$ subspaces, $X_n = \{a,b\} \cup \bigcup_{i=1}^{\infty} A_i \cup \bigcup_{j=1}^{\infty} B_j \cup \bigcup_{k=1}^{n} C_k$.

The next (modified) example from Dugundji [2, p. 422] shows that if the bonding maps are not one-to-one, even the best of spaces have bad direct limits.

Example 3.12

For each nonnegative integer n, let X_n be the unit circle S^1 in the complex plane. Let $f_m^n = z^{2^{n-m}}$. Since for each n, f_0^n is an epimorphism, the injection u_0 of X_0 into $\varinjlim X_n$ is an epimorphism. Given any point $e^{i\alpha}$ in X_0, it is clear that $A_\alpha = \{e^{i(\frac{m}{2^n}\pi + \alpha)}: m$ and n are positive integers$\}$ is dense in X_0 and contains $e^{i\alpha}$. If $e^{i(\frac{m_1}{2^{n_1}}\pi+\alpha)}$ and $e^{i(\frac{m_2}{2^{n_2}}\pi+\alpha)}$ are any two elements of this set, then $f_0^{n_1+n_2}(e^{i(\frac{m_1}{2^{n_1}}\pi+\alpha)}) = e^{i(2^{n_2}m_1\pi+2^{n_1+n_2}\alpha)} = e^{i(2^{n_1}m_2\pi+2^{n_1+n_2}\alpha)} =$ $f_0^{n_1+n_2}(e^{i(\frac{m_2}{2^{n_2}}\pi+\alpha)})$. Hence, u_0 identifies all points of A_α. Then given any nonempty open set U in $\varinjlim X_n$, since $u_0^{-1}[U]$ is a nonempty open set in X_0, $u_0^{-1}[U] \cap A_\alpha \neq \emptyset$; so $u_0(e^{i\alpha}) \in U$. Therefore, because this is true of any $e^{i\alpha}$ and every nonempty open set in $\varinjlim X_n$, the only nonempty open set in $\varinjlim X_n$ is the whole space. Thus, $\varinjlim X_n$ is indiscrete. Also if π/α is not rational, then for each $m \geq 0$, $f_0^m(e^{i\alpha}) = e^{i2^m\alpha} \neq e^{i\cdot 0} = f_0^m(1)$; so that $\varinjlim X_n$ contains more than one point.

The next example shows that in Theorem 3.10 the condition that the bonding maps be relatively open cannot be deleted.

Example 3.13

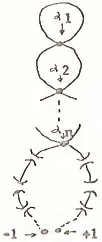
Select an increasing sequence $\{r_n\}$ of points in the closed unit interval $[0,1]$ which converges to 1. Let $X = ([0,1] \sim \{r_n\}) \cup \{\alpha\}$. For each integer, n, let X_n be the space whose set is X but whose topology is generated by the relative topology on $[0,1] \sim \{r_n\}$ and by sets of the form $(\{\alpha\} \cup A) \cap X$ where A is an open subset of $[0,1]$ containing $\{r_1, r_2,,, r_n\}$. Clearly, for each $n \geq m$, the identity function on X is a continuous function i_m^n from X_m to X_n and $\{X_n, i_m^n, N\}$ is a direct system of Hausdorff spaces. For each $n \in N$, let i_n denote the canonical map from X_n to $\varinjlim X_n$. Since each i_m^n is one-to-one, each i_n is one-to-one. If U and V are open subsets of $\varinjlim X_n$ containing 1 and α, respectively, then it is clear that $U \cap ([0,1] \sim \{r_i\})$ contains a set of the form $(a,1] \cap ([0,1] \sim \{r_i\})$ for some $a < 1$. Hence, there is some k such that $r_k > a$. Since $i_k^{-1}[V]$ is open in X_k, $(a,1] \cap i_k^{-1}[V]$ is nonempty, so $U \cap V$ is nonempty; hence, $\varinjlim X_n$ is not Hausdorff.

Next, we give an example to show that certain more stringent separation properties are not preserved under the conditions of Theorem 3.10.

Example 3.14

Select an increasing sequence of points $\{r_n\}$ from the open interval (0,1) which converges to 1. Let X be $[-1,1] \sim \{\pm r_i\}$. (I.e., the closed interval with two sequences deleted, one increasing to 1 and the other decreasing to -1.) For each $n \in N$, let X_n be $X \cup \{\alpha_1, \alpha_2,,, \alpha_n\}$ with topology generated by the

relative topology on X and by sets of the form $\{\alpha_i\} \cup (V \cap X)$ where V is an open subset of $[0,1]$ which contains both r_i and $-r_i$. Hence, X_n will be metrizable and look something like



Together with the inclusion functions i_m^n serving as bonding maps, $\{X_n : n \in N\}$ forms a direct system. Furthermore, each inclusion function is open, and X is a dense subset of X_n for each n. Hence, by Theorem 3.10, $\varinjlim X_n$ is Hausdorff. Denote by i_m the canonical map of X_m to $\varinjlim X_n$. If U is an open set in $\varinjlim X_n$ containing $i_m(1)$, then $i_m^{-1}[U]$ is an open subset of X_m containing 1; so from the definition of the topology on X_m, there is some interval $(a,1]$ in $[-1,1]$ such that $X \cap ((a,1]) \subseteq i_m^{-1}[U]$. Since $\{r_n\}$ converges to 1, there is some $k \in N$ such that for $n \geq k$, $r_n \in (a,1]$. Then for any open subset V in X_k containing α_k, $V \cap i_k^{-1}[U] \neq \emptyset$. $\bigcup_{j=1}^{\infty} i_j(\alpha_j)$ is closed in $\varinjlim X_n$ since for any ℓ, $i_\ell^{-1}[\bigcup_{j=1}^{\infty} i_j(\alpha_j)]$ is a finite (hence closed) subset of X_ℓ. Therefore, $\varinjlim X_n$ is not T_3, since $i_n(1)$ and $\bigcup_{j=1}^{n} i_j(\alpha_j)$ cannot be separated by open sets.

Much stronger conditions are required to preserve the regular Hausdorff property.

Theorem 3.15

If $\{X_\lambda, f_\alpha^\beta, \Lambda\}$ is a direct system of T_3 spaces such that:

(i) $\varinjlim X_\lambda$ is T_1

(ii) Each bonding map f_α^β is one-to-one and open, and

(iii) For each $x \in X_\lambda$, there is some open set U in X_λ containing x such that if A is a closed subset of X_λ contained in U, then $f_\lambda^\alpha[A]$ is closed in X_α for every $\alpha \geq \lambda$;

then $\varinjlim X_\lambda$ is T_3.

Proof: Let a be an element of $\varinjlim X_\lambda$ and let B be a closed subset of $\varinjlim X_\lambda$ which does not contain a. Then there is some $x \in X_\lambda$ such that a is the image of x under the map i_λ. Since i_λ is continuous, $i_\lambda^{-1}[B]$ is a closed subset of X_λ which does not contain x. Let U be an open neighborhood of x as described in condition (iii) of the theorem. Since $(X \sim U) \cup i_\lambda^{-1}[B]$ is a closed subset of X_λ which does not contain x, and since X_λ is a T_3 space, there is an open subset V of X_λ such that $x \in V \subseteq \overline{V} \subseteq X \sim ((X \sim U) \cup i_\lambda^{-1}[B])$. Then, since $V \subseteq \overline{V} \subseteq U$, for any $\alpha \geq \lambda$, $f_\lambda^\alpha[V] \subseteq \overline{f_\lambda^\alpha[V]} = f_\lambda^\alpha[\overline{V}]$. Since each f_β^γ is one-to-one, each i_γ is one-to-one, and so, because each f_β^γ is open, $i_\beta^{-1}[i_\lambda[V]] = (f_\beta^\alpha)^{-1}[f_\lambda^\alpha[V]]$ is open in X_β $(\beta \leq \alpha \geq \lambda)$. Hence, $i_\lambda[V]$ is open in $\varinjlim X_\lambda$. Since for each $\alpha \geq \lambda$, $i^{-1}[i_\lambda[\overline{V}]] = (f_\beta^\alpha)^{-1}[f_\lambda^\alpha[\overline{V}]]$ is closed in X_β $(\beta \leq \alpha \geq \lambda)$, it follows that $i_\lambda[\overline{V}]$ is closed in $\varinjlim X_\lambda$. Then $i_\lambda(x) = a \in i_\lambda[V] \subseteq i_\lambda[\overline{V}] \subseteq \varinjlim X_\lambda \sim B$.

BIBLIOGRAPHY

[1]. R. M. Brooks, On Wallman Compactifications, Fundamenta Mathematicae, 60 (1967) 157-173.

[2]. J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.

[3]. H. Herrlich, Separation Axioms and Direct Limits, Canadian Math. Bull., 12 (1969) 337-338.

[4]. H. Herrlich and G. Strecker, Category Theory, Allyn and Bacon, Boston, To Appear.

[5]. J. L. Kelley, General Topology, Van Nostrand, New York, 1955.

[6]. D. Strauss, Extremally Disconnected Spaces, Proc. Amer. Math. Soc., 18 (1967) 305-309.

[7]. H. Wallman, Lattices and Topological Spaces, Annals of Math., 39 (1938) 112-126.

BIOGRAPHICAL SKETCH

Darrell Wayne Hajek was born December 11, 1938, at Loup City, Nebraska. In June, 1957, he graduated from Loup City High School. He attended the University of Nebraska from 1957 to 1961, when he enlisted in the United States Air Force. Upon discharge in 1965, he returned to the University of Nebraska where, in June, 1966, he received the degree of Bachelor of Arts with a major in mathematics. He then attended the Graduate School of the University of Nebraska, working as a teaching assistant until June, 1968, when he received his Master of Arts degree with a major in mathematics. From September, 1968, until the present time he has been attending the Graduate School of the University of Florida on a National Science Foundation Traineeship, working on the degree of Doctor of Philosophy.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

George E. Strecker

George E. Strecker, Chairman
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Ludvik Janos

Ludvik Janos
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

William T. England

William T. England
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

David A. Drake

David A. Drake
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

David P. Stadtlander

David P. Stadtlander
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

William H. Boykin, Jr.

William H. Boykin, Jr.
Assistant Professor of Engineering Science and Mechanics

This dissertation was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August, 1971

C.A. Vanderwerf

Dean, College of Arts and Sciences

Dean, Graduate School